AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname ______________________

Forename(s) ______________________

Candidate signature ______________________
I declare this is my own work.

A-level
FURTHER MATHEMATICS
Paper 2

Monday 5 June 2023 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a graphical or scientific calculator that meets the requirements of the specification.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
TOTAL	

Do not write outside the box

Answer **all** questions in the spaces provided.

1 Given that $y = \sin x + \sinh x$, find $\dfrac{d^2y}{dx^2} + y$

Circle your answer.

[1 mark]

$2\sin x$ $\quad$ $-2\sin x$ $\quad$ $2\sinh x$ $\quad$ $-2\sinh x$

2 Which one of the expressions below is **not** equal to zero?

Circle your answer.

[1 mark]

$\lim_{x \to \infty}(x^2 e^{-x})$ $\quad$ $\lim_{x \to 0}(x^5 \ln x)$ $\quad$ $\lim_{x \to \infty}\left(\dfrac{e^x}{x^5}\right)$ $\quad$ $\lim_{x \to 0}(x^3 e^x)$

Do not write outside the box

3 The determinant $A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 2 & 1 \end{vmatrix}$

Which one of the determinants below has a value which is **not** equal to the value of A?

Tick (✓) **one** box.

[1 mark]

$\begin{vmatrix} 3 & 1 & 3 \\ 2 & 0 & 2 \\ 3 & 2 & 1 \end{vmatrix}$ ☐

$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{vmatrix}$ ☐

$\begin{vmatrix} 2 & 2 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix}$ ☐

$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 0 & 2 \end{vmatrix}$ ☐

4 It is given that $f(x) = \cosh^{-1}(x - 3)$

Which of the sets listed below is the greatest possible domain of the function f?

Circle your answer.

[1 mark]

$\{x : x \geq 4\}$ $\{x : x \geq 3\}$ $\{x : x \geq 1\}$ $\{x : x \geq 0\}$

Do not write outside the box

5 Josh and Zoe are solving the following mathematics problem:

> The curve C_1 has equation
>
> $$\frac{x^2}{16} - \frac{y^2}{9} = 1$$
>
> The matrix $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ maps C_1 onto C_2
>
> Find the equations of the asymptotes of C_2

Josh says that to solve this problem you **must** first carry out the transformation on C_1 to find C_2, and then find the asymptotes of C_2

Zoe says that you will get the same answer if you first find the asymptotes of C_1, and then carry out the transformation on these asymptotes to obtain the asymptotes of C_2

Show that Zoe is correct.

[5 marks]

6 (a) Express $-5 - 5\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leq \pi$

[2 marks]

6 (b) The point on an Argand diagram that represents $-5 - 5\mathrm{i}$ is one of the vertices of an equilateral triangle whose centre is at the origin.

Find the complex numbers represented by the other two vertices of the triangle.

Give your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leq \pi$

[3 marks]

Do not write outside the box

7 Show that

$$\sum_{r=11}^{n+1} r^3 = \frac{1}{4}(n^2 + an + b)(n^2 + an + c)$$

where a, b and c are integers to be found.

[3 marks]

Do not write outside the box

8 **A** is a non-singular 2×2 matrix and $\mathbf{A}^{\mathrm{T}}$ is the transpose of **A**

8 (a) Using the result

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

show that

$$(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1}$$

[3 marks]

8 (b) It is given that $\mathbf{A} = \begin{bmatrix} 4 & 5 \\ -1 & k \end{bmatrix}$, where k is a real constant.

8 (b) (i) Find $(\mathbf{A}^{-1})^{\mathrm{T}}$, giving your answer in terms of k

[2 marks]

8 (b) (ii) State the restriction on the possible values of k

[1 mark]

Turn over ▶

Do not write outside the box

9 The complex number z is such that

$$z = \frac{1 + \mathrm{i}}{1 - k\mathrm{i}}$$

where k is a real number.

9 (a) Find the real part of z and the imaginary part of z, giving your answers in terms of k

[2 marks]

9 (b) In the case where $k = \sqrt{3}$, use part **(a)** to show that

$$\cos \frac{7\pi}{12} = \frac{\sqrt{2} - \sqrt{6}}{4}$$

[5 marks]

Do not write outside the box

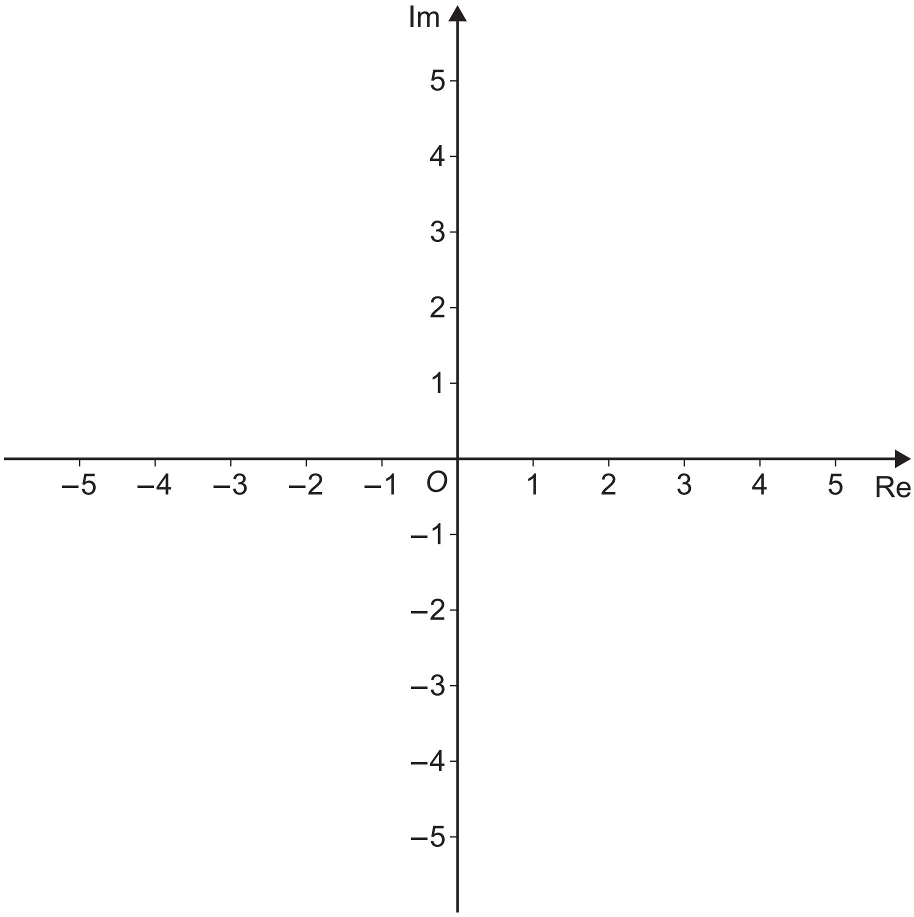
Turn over ▶

Do not write outside the box

10 The region R on an Argand diagram satisfies both $|z + 2\mathrm{i}| \leq 3$ and $-\dfrac{\pi}{6} \leq \arg(z) \leq \dfrac{\pi}{2}$

10 (a) Sketch R on the Argand diagram below.

[3 marks]



Do not write outside the box

10 (b) Find the maximum value of $|z|$ in the region R, giving your answer in exact form.

[5 marks]

Turn over for the next question

Do not write outside the box

11 The line l_1 passes through the points $A(6, 2, 7)$ and $B(4, -3, 7)$

11 (a) Find a Cartesian equation of l_1

[2 marks]

11 (b) The line l_2 has vector equation $\mathbf{r} = \begin{bmatrix} 8 \\ 9 \\ c \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ where c is a constant.

11 (b) (i) Explain how you know that the lines l_1 and l_2 are not perpendicular.

[2 marks]

Do not write outside the box

11 (b) (ii) The lines l_1 and l_2 both lie in the same plane.

Find the value of c

[5 marks]

Turn over for the next question

12 The function f is defined by

$$f(n) = 3^{3n+1} + 2^{3n+4} \qquad (n \in \mathbb{Z}^+)$$

Prove by induction that $f(n)$ is divisible by 19 for $n \geq 1$

[6 marks]

Do not write outside the box

Turn over for the next question

Do not write outside the box

13 The quadratic equation $z^2 - 5z + 8 = 0$ has roots α and β

13 (a) Write down the value of $\alpha + \beta$ and the value of $\alpha\beta$

[2 marks]

13 (b) Without finding the value of α or the value of β, show that $\alpha^4 + \beta^4 = -47$

[4 marks]

Do not write outside the box

13 (c) Find a quadratic equation, with integer coefficients, which has roots $\alpha^3 + \beta$ and $\beta^3 + \alpha$

[5 marks]

Turn over for the next question

Do not write outside the box

14 The function f is defined by

$$\mathrm{f}(x) = \frac{1}{4x^2 + 16x + 19} \qquad (x \in \mathbb{R})$$

14 (a) Show, without using calculus, that the graph of $y = \mathrm{f}(x)$ has a stationary point at $\left(-2, \frac{1}{3}\right)$

[3 marks]

14 (b) Show that $\int_{-2}^{-\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x = \frac{\pi\sqrt{3}}{18}$

[5 marks]

Do not write outside the box

14 (c) Find the value of $\int_{-2}^{\infty} \mathrm{f}(x)\,\mathrm{d}x$

Fully justify your answer.

[2 marks]

Do not write outside the box

15 (a) Given that $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$z^n - z^{-n} = 2\mathrm{i}\sin n\theta$$

[2 marks]

15 (b) The series S is defined as

$$S = \sin\theta + \sin 3\theta + ... + \sin(2n-1)\theta$$

Use part **(a)** to express S in the form

$$S = \frac{1}{2\mathrm{i}}(G_1) - \frac{1}{2\mathrm{i}}(G_2)$$

where each of G_1 and G_2 is a geometric series.

[3 marks]

Do not write outside the box

15 (c) Hence, show that

$$S = \frac{\sin^2 (n\theta)}{\sin \theta}$$

[5 marks]

Do not write outside the box

16 A bungee jumper of mass m kg is attached to an elastic rope.
The other end of the rope is attached to a fixed point.

The bungee jumper falls vertically from the fixed point.

At time t seconds after the rope first becomes taut, the extension of the rope is x metres and the speed of the bungee jumper is $v\,\text{m s}^{-1}$

16 (a) A model for the motion while the rope remains taut assumes that the forces acting on the bungee jumper are

- the weight of the bungee jumper
- a tension in the rope of magnitude kx newtons
- an air resistance force of magnitude Rv newtons

where k and R are constants such that $4km > R^2$

16 (a) (i) Show that this model gives the result

$$x = \mathrm{e}^{-\frac{Rt}{2m}}\left(A\cos\left(\frac{\sqrt{4km - R^2}}{2m}\right)t + B\sin\left(\frac{\sqrt{4km - R^2}}{2m}\right)t\right) + \frac{mg}{k}$$

where A and B are constants, and $g\,\text{m s}^{-2}$ is the acceleration due to gravity.

You do not need to find the value of A or the value of B

[6 marks]

Do not write outside the box

16 (a) (ii) It is also given that:

$$k = 16$$
$$R = 20$$
$$m = 62.5$$
$$g = 9.8\,\text{m s}^{-2}$$

and that the speed of the bungee jumper when the rope becomes taut is $14\,\text{m s}^{-1}$

Show that, to the nearest integer, $A = -38$ and $B = 16$

[6 marks]

Do not write outside the box

16 (b) A second, simpler model assumes that the air resistance is zero.

The values of k, m and g remain the same.

Find an expression for x in terms of t according to this simpler model, giving the values of all constants to two significant figures.

[4 marks]

END OF QUESTIONS

Do not write outside the box

There are no questions printed on this page

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Do not write outside the box

Question number	**Additional page, if required. Write the question numbers in the left-hand margin.**

Do not write outside the box

Question number	**Additional page, if required. Write the question numbers in the left-hand margin.**