A-level
FURTHER MATHEMATICS
7367/2

Paper 2

Mark scheme

June 2021

Version: 1.0 Final Mark Scheme

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this mark scheme are available from aqa.org.uk

Mark scheme instructions to examiners

General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

Key to mark types

M	mark is for method
R	mark is for reasoning
A	mark is dependent on M marks and is for accuracy
B	mark is independent of M marks and is for method and accuracy
E	mark is for explanation
F	follow through from previous incorrect result

Key to mark scheme abbreviations

CAO	correct answer only
CSO	correct solution only
ft	follow through from previous incorrect result
'their'	indicates that credit can be given from previous incorrect result
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
NMS	no method shown
PI	possibly implied
sf	significant figure(s)
dp	decimal place(s)

Examiners should consistently apply the following general marking principles

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Diagrams

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

Work erased or crossed out

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

Choice

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, mark positively, awarding marks for all of the student's best attempts. Withhold marks for final accuracy and conclusions if there are conflicting complete answers or when an incorrect solution (or part thereof) is referred to in the final answer.

AS/A-level Maths/Further Maths assessment objectives

AO		Description
AO1	AO1.1a	Select routine procedures
	AO1.1b	Correctly carry out routine procedures
	AO1.2	Accurately recall facts, terminology and definitions
AO2	AO2.1	Construct rigorous mathematical arguments (including proofs)
	AO2.2a	Make deductions
	AO2.2b	Make inferences
	AO2.3	Assess the validity of mathematical arguments
	AO2.4	Explain their reasoning
	AO2.5	Use mathematical language and notation correctly
AO3	AO3.1a	Translate problems in mathematical contexts into mathematical processes
	AO3.1b	Translate problems in non-mathematical contexts into mathematical processes
	AO3.2a	Interpret solutions to problems in their original context
	AO3.2b	Where appropriate, evaluate the accuracy and limitations of solutions to problems
	AO3.3	Translate situations in context into mathematical models
	AO3.4	Use mathematical models
	AO3.5a	Evaluate the outcomes of modelling in context
	AO3.5b	Recognise the limitations of models
	AO3.5c	Where appropriate, explain how to refine models

Q	Marking Instructions	AO	Marks	Typical solution
1	Circles correct answer	1.1b	B1	$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
2	Circles correct answer	1.1b	B1	$-120°$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
3	Ticks correct answer	2.2a	B1	$\mathbf{r} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
4(a)	Completes a rigorous argument to show the required result Must begin with $(r+1)^2 - r^2 = \ldots$	2.1	R1	$(r+1)^2 - r^2 = r^2 + 2r + 1 - r^2$ $= 2r + 1$ as required
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
4(b)	Uses method of differences including at least the first two or last two terms	1.1a	M1	$\sum_{r=1}^{n}(2r+1) = \sum_{r=1}^{n}((r+1)^2 - r^2)$ $= \cancel{2^2} - 1^2$ $+\cancel{3^2} - \cancel{2^2}$ $+\cdots$ $+\cdots$ $+\cancel{n^2} - \cancel{(n-1)^2}$ $+(n+1)^2 - \cancel{n^2}$ $= (n+1)^2 - 1$ $= n^2 + 2n + 1 - 1$ $= n^2 + 2n$
	Identifies and simplifies the two remaining terms	1.1a	M1	
	Completes a rigorous argument to show the required result, including seeing at least the first two and the last two terms. Must begin with $\sum_{r=1}^{n}(2r+1) = \cdots$	2.1	R1	
	Total		**3**	

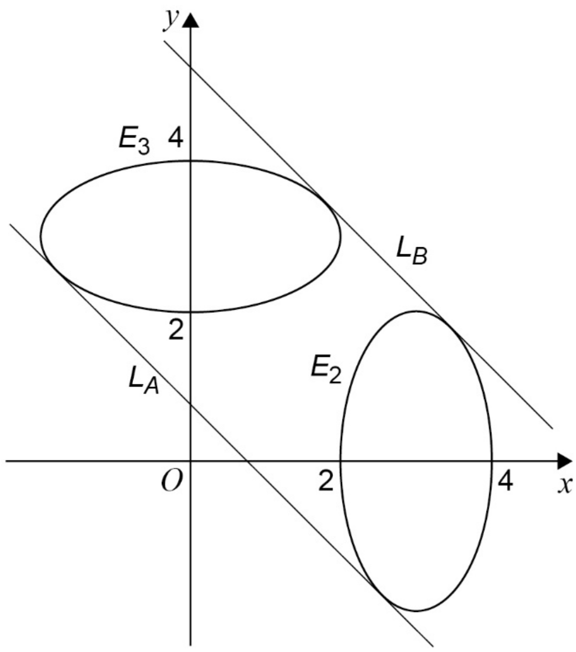
Q	Marking Instructions	AO	Marks	Typical solution
4(c)	Recalls and states $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$	1.2	B1	$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ $\therefore \sum_{r=1}^{n}(2r+1) = 2 \times \frac{1}{2}n(n+1) + n$ $= n^2 + n + n$ $= n^2 + 2n$ as required
	Splits the sum into two parts and uses their formula.	1.1a	M1	
	Completes a clear argument to show the required result. Condone the lack of limits on the summation signs. Must begin with $\sum_{r=1}^{n}(2r+1) = \cdots$	2.1	R1	
	Total		**3**	

	Question total		**7**	

Q	Marking Instructions	AO	Marks	Typical solution
5	Expresses w in terms of z or States the values of the sum, product and pairwise sum of roots of original equation. Condone sign errors.	3.1a	M1	$w = \frac{1}{2}z - 1$ so $z = 2w + 2$ Substituting in the original equation $(2w+2)^3 + 2(2w+2)^2 - 5(2w+2) - 3 = 0$ $8w^3 + 24w^2 + 24w + 8 + 8w^2 + 16w + 8$ $- 10w - 10 - 3 = 0$ $8w^3 + 32w^2 + 30w + 3 = 0$
	Expresses z correctly in terms of w or obtains correct value (-4) of sum of roots of new equation	1.1b	A1	
	Substitutes their expression for z into original equation to form an equation in w or Expresses the pairwise sum of roots of new equation in terms $\sum \alpha$ and $\sum \alpha\beta$	1.1a	M1	
	Simplifies their equation in w or Expresses product of roots of new equation in terms $\sum \alpha$, $\sum \alpha\beta$ and $\alpha\beta\gamma$	1.1a	M1	
	Obtains correct equation (any correct form)	1.1b	A1	
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical solution
6(a)	Obtains the correct equation of E_2	1.1b	B1	Equation of E_2 is $$(x-3)^2+\frac{y^2}{4}=1$$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
6(b)	Obtains a correct answer Condone a correct sequence of transformations	1.1b	B1	Reflection in the line $y=x$
	Total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
6(c)	Two ellipses, one crossing the positive x-axis and the other crossing the positive y-axis	1.1b	B1	
	Correct axis intercepts shown for both ellipses	1.1b	B1	
	At least one correct tangent drawn Condone x = 2 or y = 2	1.1b	B1	
	Both lines drawn and labelled correctly	2.2b	B1	
	Total		**4**	

Q	Marking Instructions	AO	Marks	Typical solution
6(d)	Uses the fact that E_3 is a reflection of E_2 in the line $y=x$	2.4	E1	Points on E_2 and E_3 joined by L_A are symmetrical about y = x, therefore the line is perpendicular to y = x and has a gradient of -1 and is of the form y = -x + c or x + y = c
	Explains that the tangent is perpendicular to the line $y=x$ and concludes that its equation is x + y = c	2.4	E1	
	Total		**2**	

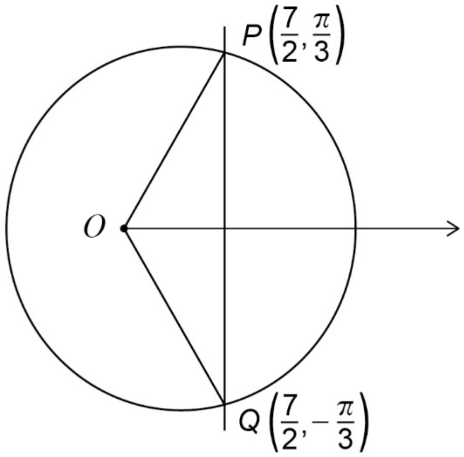
	Question total		**8**	

Q	Marking Instructions	AO	Marks	Typical Solution
7	Obtains derivatives of x and y	1.1a	M1	$\dot{x} = -12\cos^2 t \sin t$ $\dot{y} = 12\sin^2 t \cos t$ $\dot{x}^2 + \dot{y}^2 = 144\cos^4 t \sin^2 t + 144\sin^4 t \cos^2 t$ $= 144\cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)$ $= 144\cos^2 t \sin^2 t$ $\sqrt{\dot{x}^2 + \dot{y}^2} = 12\cos t \sin t$ $S = 2\pi \int_0^{\frac{\pi}{2}} y\sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}t$ $= 2\pi \int_0^{\frac{\pi}{2}} 4\sin^3 t\,(12\cos t \sin t)\mathrm{d}t$ $= 96\pi \int_0^{\frac{\pi}{2}} \sin^4 t \cos t\,\mathrm{d}t$ $= 96\pi \left[\frac{\sin^5 t}{5}\right]_0^{\frac{\pi}{2}}$ $= \frac{96\pi}{5}$
	Obtains correct expression for $\dot{x}^2 + \dot{y}^2$	1.1b	A1	
	Uses trig identity to simplify their expression for $\dot{x}^2 + \dot{y}^2$	2.2a	B1	
	Substitutes their expression for $\dot{x}^2 + \dot{y}^2$ into the formula for surface area Condone missing limits of integration and missing "2π"	1.1a	M1	
	Obtains correct expression for the surface area including correct limits of integration	1.1b	A1	
	Obtains $k\sin^5 t$ by integration Condone missing limits of integration	1.1b	A1	
	Completes a rigorous argument to show the required result	2.1	R1	
	Total		**7**	

Q	Marking Instructions	AO	Marks	Typical solution
8	Expresses z and z* in terms of x and y and substitutes these in the first equation	1.1a	M1	Let $z = x + \mathrm{i}y$ Then $\lvert x - \mathrm{i}y - 1 - 2\mathrm{i} \rvert = \lvert x + \mathrm{i}y - 3 \rvert$ $(x-1)^2 + (y+2)^2 = (x-3)^2 + y^2$ $-2x + 1 + 4y + 4 = -6x + 9$ $y = 1 - x$ and $(x-a)^2 + y^2 = 9$ Solving simultaneously, $(x-a)^2 + (1-x)^2 = 9$ $2x^2 + (-2a-2)x + (a^2 - 8) = 0$ $\Delta \geq 0$ so $(2a+2)^2 - 4(2)(a^2-8) \geq 0$ $a^2 - 2a - 17 \leq 0$ $(a-1)^2 \leq 18$ So $1 - \sqrt{18} \leq a \leq 1 + \sqrt{18}$ And a must lie in the interval $\left[1 - 3\sqrt{2}, 1 + 3\sqrt{2}\right]$
	Deduces a correct linear Cartesian equation for first equation	2.2a	A1	
	Obtains correct Cartesian equation for second equation or states that this represents a circle centre (a, 0) radius 3	1.1b	B1	
	Solves their Cartesian equations simultaneously	1.1a	M1	
	States that the discriminant is non-negative or creates an inequality based on this	2.4	E1	
	Completes a rigorous argument to obtain the correct range of values for a	2.1	R1	
	Total		**6**	

Q	Marking Instructions	AO	Marks	Typical solution
9(a)	Obtains $x = \frac{7}{4}$ oe	1.1b	B1	$r = \frac{7}{4}\sec\theta$ $r\cos\theta = \frac{7}{4}$ And $x = r\cos\theta$ So in Cartesian coordinates: $x = \frac{7}{4}$ which is perpendicular to the x-axis $\therefore$ *L* is perpendicular to the initial line
	Explains that *L* is perpendicular to the initial line or x-axis	2.1	E1	
	Total		**2**	

Q	Marking Instructions	AO	Marks	Typical solution
9(b)	Obtains equation in θ or r	1.1a	M1	At points of intersection $\frac{7}{4}\sec\theta = 3 + \cos\theta$ $\cos^2\theta + 3\cos\theta - \frac{7}{4} = 0$ $4\cos^2\theta + 12\cos\theta - 7 = 0$ $\cos\theta = -\frac{7}{2}$ (reject as -1 ≤ cosθ≤ 1) or $\cos\theta = \frac{1}{2}$ Points are $\left(\frac{7}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{7}{2}, -\frac{\pi}{3}\right)$
	Rearranges and solves for $\cos\theta$ or $\sec\theta$ or r	3.1a	M1	
	Rejects, with a reason, the impossible value of $\cos\theta$ or r	2.2a	E1	
	Obtains correct value of $\cos\theta$ or r	1.1b	A1	
	Obtains correct polar coordinates	1.1b	A1	
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical solution
9(c)	Identifies the required region.	3.1a	M1	
	Obtains correct area of triangle	1.1b	B1	
	Obtains $k\int(3+\cos\theta)^2 d\theta$ with or without limits	3.1a	M1	PQ= $\frac{7\sqrt{3}}{2}$
	Uses $\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$	3.1a	M1	Area of triangle OPQ $=\frac{1}{2}\times\frac{7\sqrt{3}}{2}\times\frac{7}{4}=\frac{49\sqrt{3}}{16}$
	Integrates their (at least three-part) expression correctly	1.1b	A1F	Area of sector OPQ $= 2\left(\int_0^{\frac{\pi}{3}}\frac{1}{2}r^2\,\mathrm{d}\,\theta\right)$ (by symmetry) $=\int_0^{\frac{\pi}{3}}(3+\cos\theta)^2\,\mathrm{d}\theta$
	Obtains correct area of sector or half of sector	1.1b	A1	$=\int_0^{\frac{\pi}{3}}(9+6\cos\theta+\cos^2\theta)\mathrm{d}\theta$
	Obtains exact correct answer	1.1b	A1	$=\int_0^{\frac{\pi}{3}}(9+6\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2})\mathrm{d}\theta$ $=[\frac{19\theta}{2}+6\sin\theta+\frac{1}{4}\sin 2\theta]_0^{\frac{\pi}{3}}$ $=\left(\frac{19\pi}{6}+3\sqrt{3}+\frac{\sqrt{3}}{8}\right)-0$ $=\frac{19\pi}{6}+\frac{25\sqrt{3}}{8}$ Required area $=\frac{19\pi}{6}+\frac{25\sqrt{3}}{8}-\frac{49\sqrt{3}}{16}$ $=\frac{19\pi}{6}+\frac{\sqrt{3}}{16}$
	Total		**7**	

	Question total		**14**	

Q	Marking Instructions	AO	Marks	Typical solution
10(a)	Separates the variables	1.1a	M1	$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky$ $\int \frac{1}{y}\mathrm{d}y = \int -k\mathrm{d}t$ So $y = y_0\mathrm{e}^{-kt}$ $0.84y_0 = y_0\mathrm{e}^{-k}$ $k = 0.174$
	Deduces exponential form	2.2a	M1	
	Forms correct equation	3.3	M1	
	Obtains correct answer	1.1b	A1	
	Total		**4**	

Q	Marking Instructions	AO	Marks	Typical solution
10(b)	Forms DE with three terms and $\frac{\mathrm{d}y}{\mathrm{d}t}$	3.3	M1	$\frac{\mathrm{d}y}{\mathrm{d}t} = -0.174y + 45 + 20t$
	Obtains correct DE (consistent with their answer to part (a)) may use k in place of 0.174	3.3	A1F	
	Total		**2**	

Q	Marking Instructions	AO	Marks	Typical solution
10(c)	Splits into CF and PI Or Finds an integrating factor	3.1a	M1	$\frac{\mathrm{d}y}{\mathrm{d}t} + 0.174y = 45 + 20t$ CF: $y = A\mathrm{e}^{-0.174t}$ PI: $y = p + qt$ $\dot{y} = q$ $q + 0.174p + 0.174qt = 45 + 20t$ $q = \frac{20}{0.174} = 115$ $p = \frac{45 - q}{0.174} = -402$ General solution: $y = A\mathrm{e}^{-0.174t} - 402 + 115t$ $t = 0 \Rightarrow y = 340$ So $A = 742$ $y = 742\mathrm{e}^{-0.174t} - 402 + 115t$
	Obtains PI of the form $p + qt$ Or Uses integration by parts to integrate the te^{kt} term	1.1a	M1	
	Obtains correct general solution from their value of k	1.1b	A1F	
	Uses initial conditions to find the unknown constant for a general solution that includes an exponential term	3.3	M1	
	Obtains correct solution	1.1b	A1	
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical solution
10(d)	One reasonable limitation of their model for limitation 1	3.5b	E1	Births would occur at a particular time of year, not at a steady rate
	One reasonable limitation of their model for limitation 2	3.5b	E1	Over a long period of time the population would increase indefinitely according to the model
	Total		**2**	

	Question total		**13**	

Q	Marking Instructions	AO	Marks	Typical solution
11	Obtains a position vector of a point on L_1 or L_2	2.5	B1	
	Obtains a direction vector for L_1 or L_2, ISW	1.1b	B1	The position vector of a point on L_1 is $\begin{bmatrix} -1 \\ 5 \\ -5/2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -2 \\ 3/2 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 5 \\ -5/2 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$
	Obtains vector equations for both L_1 and L_2	2.5	B1	The position vector of a point on L_2 is $\begin{bmatrix} 1/2 \\ 14 \\ -12 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ m \\ p \end{bmatrix}$
	Forms a matrix equation equating the image of a general point on L_1 with a general point on L_2 Condone same parameter used twice	3.1a	M1	$\begin{bmatrix} 0.5+\mu \\ 14+m\mu \\ -12+p\mu \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 & 2 \\ 1 & b & 4 \\ -3 & -2 & c \end{bmatrix} \begin{bmatrix} -1+6\lambda \\ 5-4\lambda \\ -2.5+3\lambda \end{bmatrix}$
	Collects and simplifies terms	1.1b	M1	$= \begin{bmatrix} 0.5-3\lambda+5-4\lambda-5+6\lambda \\ -1+6\lambda+5b-4b\lambda-10+12\lambda \\ 3-18\lambda-10+8\lambda-2.5c+3c\lambda \end{bmatrix}$ $= \begin{bmatrix} 0.5-\lambda \\ 5b-11+\lambda(18-4b) \\ -7-2.5c+\lambda(3c-10) \end{bmatrix}$
	Compares their constant terms to obtain a value for at least one of b or c Must have used different parameters for their general points	3.1a	M1	$0.5+\mu = 0.5-\lambda$ Therefore, when $\mu = 0$, $\lambda = 0$ so we can equate the constant terms in the equation
	Obtains correct values of both b and c	1.1b	A1	$\left.\begin{array}{l} 14 = -11+5b \\ -12 = -7-5c/2 \end{array}\right\} \rightarrow \begin{array}{l} b = 5 \\ c = 2 \end{array}$
	Uses their b and c to obtain a value for at least one of m or p	2.2a	M1	$\left.\begin{array}{l} \mu = -\lambda \\ m\mu = -2\lambda \\ p\mu = -4\lambda \end{array}\right\} \rightarrow \begin{array}{l} m = 2 \\ p = 4 \end{array}$
	Obtains correct values of m and p	1.1b	A1	
	Total		**9**	

Q	Marking Instructions	AO	Marks	Typical Solution
12(a)	Selects a method to find the required result by integrating by parts	1.1a	M1	$S_n = \int_0^a x^n \sinh x \, dx$ $u = x^n \quad v' = \sinh x$ $u' = nx^{n-1} \quad v = \cosh x$ $S_n = [x^n \cosh x]_0^a - \int_0^a nx^{n-1} \cosh x \, dx$ $= a^n \cosh a - n\int_0^a x^{n-1} \cosh x \, dx$ $u = x^{n-1} \quad v' = \cosh x$ $u' = (n-1)x^{n-2} \quad v = \sinh x$ $S_n = a^n \cosh a - n\left(\left[x^{n-1} \sinh x\right]_0^a - \int_0^a (n-1)x^{n-2} \sinh x dx\right)$ $S_n = a^n \cosh a - n(a^{n-1} \sinh a - (n-1)S_{n-2})$ $S_n = a^n \cosh a - na^{n-1} \sinh a + n(n-1)S_{n-2}$ $S_n = n(n-1)S_{n-2} + a^n \cosh a - n\, a^{n-1} \sinh a$
	Obtains the correct expressions for u' and v when integrating the first time	1.1b	A1	
	Correctly applies integration by parts formula the first time	1.1b	A1	
	Correctly applies integration by parts formula to an integral of the form $\int_0^a x^r \cosh x \, dx$	3.1a	M1	
	Obtains correct result from 2nd integration by parts and substitutes limits correctly in all terms	1.1b	A1	
	Obtains an expression for S_n in terms of S_{n-2}	1.1a	M1	
	Completes a rigorous argument to show the required result, including correct use of limits throughout	2.1	R1	
	Total		**7**	

Q	Marking Instructions	AO	Marks	Typical Solution
12(b)	Deduces that the integral is S_4 with a =1 and requires S_2 and S_0	2.2a	M1	The required integral is S_4 with $a = 1$ $S_0 = \int_0^1 \sinh x \, dx = [\cosh x]_0^1$ $= \cosh 1 - 1$ $S_2 = (2)(1)S_0 + \cosh 1 - 2\sinh 1$ $= 2(\cosh 1 - 1) + \cosh 1 - 2\sinh 1$ $= 3\cosh 1 - 2\sinh 1 - 2$ $S_4 = (4)(3)S_2 + \cosh 1 - 4\sinh 1$ $= 12(3\cosh 1 - 2\sinh 1 - 2) + \cosh 1$ $- 4\sinh 1$ $= 37\cosh 1 - 28\sinh 1 - 24$ $= \frac{37}{2}(e + e^{-1}) - 14(e - e^{-1}) - 24$ $= \frac{9}{2}e + \frac{65}{2}e^{-1} - 24$
	Uses the reduction formula once	3.1a	M1	
	Uses the reduction formula a second time and finds S_0	3.1a	M1	
	Converts both hyperbolic expressions to exponentials	1.1a	M1	
	Completes a rigorous argument to show the required result	2.1	R1	
	Total		**5**	

	Question total		**12**	

Q	Marking Instructions	AO	Marks	Typical solution
13(a)	Obtains at least two correct solutions	1.1a	M1	$6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$
	Obtains all correct solutions	1.1b	A1	$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$ (in addition to given solutions)
	Total		**2**	

Q	Marking Instructions	AO	Marks	Typical solution
13(b)	Expands $(\cos\theta + \mathrm{i}\sin\theta)^6$	1.1a	M1	By de Moivre's theorem $\cos 6\theta + \mathrm{i}\sin 6\theta = (\cos\theta + \mathrm{i}\sin\theta)^6$ $= \cos^6\theta + 6\cos^5\theta\,(\mathrm{i}\sin\theta) + 15\cos^4\theta\,(\mathrm{i}\sin\theta)^2 + 20\cos^3\theta(\mathrm{i}\sin\theta)^3 + 15\cos^2\theta\,(\mathrm{i}\sin\theta)^4 + 6\cos\theta\,(\mathrm{i}\sin\theta)^5 + (\mathrm{i}\sin\theta)^6$
	Equates real parts	3.1a	M1	Equating real parts $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$
	Obtains correct expression for real part in terms of powers of $\cos\theta$ and $\sin\theta$	1.1b	A1	$= \cos^6\theta - 15\cos^4\theta\,(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$
	Uses trig identity to express real part in terms of $\cos\theta$	3.1a	M1	$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$
	Completes a rigorous argument to obtain the required result	2.1	R1	$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical solution
13(c)	Uses the fact that either $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ is a solution to the first equation to deduce that it is also a solution to the second equation	2.2a	M1	$\theta = \dfrac{\pi}{4} \Rightarrow \cos 6\theta = 0$ $\Rightarrow 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = 0$ from part (b) $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\therefore \left(\cos\theta - \frac{1}{\sqrt{2}}\right)$ is a factor of $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$
	Uses the factor theorem	3.1a	M1	Similarly $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\left(\cos\theta + \frac{1}{\sqrt{2}}\right)$ is also a factor of the expression.
	Multiplies their linear factors together	1.1a	M1	So $\left(\cos\theta - \frac{1}{\sqrt{2}}\right)\left(\cos\theta + \frac{1}{\sqrt{2}}\right) = \left(\cos^2\theta - \frac{1}{2}\right)$ is a factor and $(2\cos^2\theta - 1)$ is a factor
	Obtains the correct result (oe)	2.1	R1	
	Total		**4**	

Q	Marking Instructions	AO	Marks	Typical solution
13(d)	Divides the polynomial by their quadratic factor	3.1a	M1	Let $c = \cos\theta$ $$32c^6 - 48c^4 + 18c^2 - 1 = (2c^2 - 1)(16c^4 - 16c^2 + 1)$$
	Solves their quartic equation as a quadratic in c^2	1.1a	M1	For $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$, $2c^2 - 1 = 0$ So for the other four roots, which are the cosines of the other four solutions to $\cos 6\theta = 0$, $$16c^4 - 16c^2 + 1 = 0$$
	Explains that the roots of the quartic correspond to the cosines of the angles found in part (a)	2.4	E1	Solving this as a quadratic in c^2 gives $$c^2 = \frac{2 \pm \sqrt{3}}{4}$$ So
	Obtains all correct roots of the quartic	1.1b	A1	$$c = \pm\sqrt{\frac{2 \pm \sqrt{3}}{4}}$$
	Uses a rigorous argument to obtain the required result, including a reason why that particular root corresponds to $\cos\left(\frac{11\pi}{12}\right)$	2.1	R1	Of the angles $\frac{\pi}{12}$, $\frac{5\pi}{12}$, $\frac{7\pi}{12}$, and $\frac{11\pi}{12}$, $\frac{11\pi}{12}$ has the negative cosine of the greatest magnitude Therefore $\cos\left(\frac{11\pi}{12}\right) = -\sqrt{\frac{2+\sqrt{3}}{4}}$
	Total		**5**	

	Question total		**16**	

	Paper total		**100**	