A-level

FURTHER MATHEMATICS
7367/1

Paper 1

Mark scheme

June 2024

Version: 1.0 Final

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Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

No student should be disadvantaged on the basis of their gender identity and/or how they refer to the gender identity of others in their exam responses.

A consistent use of 'they/them' as a singular and pronouns beyond 'she/her' or 'he/him' will be credited in exam responses in line with existing mark scheme criteria.

Further copies of this mark scheme are available from aqa.org.uk

Mark scheme instructions to examiners

General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

Key to mark types

M	mark is for method
R	mark is for reasoning
A	mark is dependent on M marks and is for accuracy
B	mark is independent of M marks and is for method and accuracy
E	mark is for explanation
F	follow through from previous incorrect result

Key to mark scheme abbreviations

CAO	correct answer only
CSO	correct solution only
ft	follow through from previous incorrect result
'their'	indicates that credit can be given from previous incorrect result
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
NMS	no method shown
PI	possibly implied
sf	significant figure(s)
dp	decimal place(s)
ISW	Ignore Subsequent Workings

Examiners should consistently apply the following general marking principles:

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Diagrams

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

Work erased or crossed out

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

Choice

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, mark positively, awarding marks for all of the student's best attempts. Withhold marks for final accuracy and conclusions if there are conflicting complete answers or when an incorrect solution (or part thereof) is referred to in the final answer.

AS/A-level Maths/Further Maths assessment objectives

AO		Description
AO1	AO1.1a	Select routine procedures
	AO1.1b	Correctly carry out routine procedures
	AO1.2	Accurately recall facts, terminology and definitions
AO2	AO2.1	Construct rigorous mathematical arguments (including proofs)
	AO2.2a	Make deductions
	AO2.2b	Make inferences
	AO2.3	Assess the validity of mathematical arguments
	AO2.4	Explain their reasoning
	AO2.5	Use mathematical language and notation correctly
AO3	AO3.1a	Translate problems in mathematical contexts into mathematical processes
	AO3.1b	Translate problems in non-mathematical contexts into mathematical processes
	AO3.2a	Interpret solutions to problems in their original context
	AO3.2b	Where appropriate, evaluate the accuracy and limitations of solutions to problems
	AO3.3	Translate situations in context into mathematical models
	AO3.4	Use mathematical models
	AO3.5a	Evaluate the outcomes of modelling in context
	AO3.5b	Recognise the limitations of models
	AO3.5c	Where appropriate, explain how to refine models

Q	Marking instructions	AO	Marks	Typical solution
1	Circles 2nd answer	1.1b	B1	$-\frac{1}{5}$
	Question total		**1**	

Q	Marking instructions	AO	Marks	Typical solution
2	Circles 3rd answer	2.2a	B1	z^6
	Question total		**1**	

Q	Marking instructions	AO	Marks	Typical solution
3	Circles 2nd answer	2.2a	B1	$\frac{4}{3}$
	Question total		**1**	

Q	Marking instructions	AO	Marks	Typical solution
4	Ticks 1st box	1.2	B1	$\lim_{x\to 0}\left(x^2 \ln x\right) = 0$
	Question total		**1**	

Q	Marking instructions	AO	Marks	Typical solution
5(a)	Obtains two of the three vectors connecting A, B and C $\pm\begin{bmatrix}3\\-4\\5\end{bmatrix}, \pm\begin{bmatrix}7\\2\\6\end{bmatrix}, \pm\begin{bmatrix}4\\6\\1\end{bmatrix}$ Condone one incorrect element.	1.1a	M1	$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{bmatrix}3\\-4\\5\end{bmatrix}\times\begin{bmatrix}7\\2\\6\end{bmatrix}=17\begin{bmatrix}-2\\1\\2\end{bmatrix}$
	Forms the vector product of two vectors.	1.1a	M1	
	Obtains $k\begin{bmatrix}-2\\1\\2\end{bmatrix}$	1.1b	A1	
	Subtotal		**3**	

Q	Marking instructions	AO	Marks	Typical solution
5(b)	Obtains their $-2x + y + 2z = d$ where d is a number or Evaluates the scalar product of their normal vector and a point in the plane.	1.1a	M1	$-2x+y+2z=d$ $d=\begin{bmatrix}5\\3\\4\end{bmatrix}\bullet\begin{bmatrix}-2\\1\\2\end{bmatrix}=1$
	Obtains $-2x + y + 2z = 1$ OE	1.1b	A1	$-2x+y+2z=1$
	Subtotal		**2**	

	Question total		**5**	

Q	Marking instructions	AO	Marks	Typical solution
6	Shows that $u_1 = \frac{3}{2}\times 1^2 - \frac{3}{2}\times 1 + 1 = 1$	1.1b	B1	Let $n = 1$; then the formula gives $u_1 = \frac{3}{2}\times 1^2 - \frac{3}{2}\times 1 + 1 = 1$ (So the formula is true for $n = 1$) Assume the formula is true for $n = k$ Then we want to show that $u_{k+1} = \frac{3}{2}(k+1)^2 - \frac{3}{2}(k+1) + 1$ $= \frac{3}{2}(k^2 + 2k + 1 - k - 1) + 1$ $= \frac{3}{2}(k^2 + k) + 1$ But $u_{k+1} = \frac{3}{2}k^2 - \frac{3}{2}k + 1 + 3k$ $= \frac{3}{2}(k^2 + k) + 1$ as required So the formula is also true for $n = k + 1$ The formula for u_n is true for $n = 1$ If the formula is true for $n = k$, then the formula is also true for $n = k + 1$ Hence by induction $u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$ is true for all integers $n \geq 1$
	Assumes the result is true for $n = k$ and states $u_{k+1} = \frac{3}{2}k^2 - \frac{3}{2}k + 1 + 3k$ at any point	3.1a	M1	
	Completes correct working to deduce that $\frac{3}{2}k^2 - \frac{3}{2}k + 1 + 3k$ and $\frac{3}{2}(k+1)^2 - \frac{3}{2}(k+1) + 1$ are equivalent	2.2a	B1	
	Concludes a reasoned argument by stating: $u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$ true for $n = 1$ (seen anywhere) If $u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$ Is true for $n = k$, then true for $n = k + 1$ Hence (by induction) $u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$ is true for all **integers** $n \geq 1$ Condone "the result/formula" instead of $u_n = \frac{3}{2}n^2 - \frac{3}{2}n + 1$	2.1	R1	
	Question total		**4**	

Q	Marking instructions	AO	Marks	Typical solution
7	Writes $z = x + \mathrm{i}y \quad z^* = x - \mathrm{i}y$ $w = u + \mathrm{i}v \quad w^* = u - \mathrm{i}v$ OE PI or Obtains the conjugate of one of the equations Eg $z^* + w = 5$	1.1a	M1	Let $z = x + \mathrm{i}y \quad z^* = x - \mathrm{i}y$ $w = u + \mathrm{i}v \quad w^* = u - \mathrm{i}v$ where $x, y, u, v \in \mathbb{R}$ $x + \mathrm{i}y + u - \mathrm{i}v = 5$ Re: $x + u = 5$…(1) Im: $y - v = 0 \Rightarrow y = v$…(2) $3(x - \mathrm{i}y) - (u + \mathrm{i}v) = 6 + 4\mathrm{i}$ Re: $3x - u = 6$…(3) Im: $-3y - v = 4$…(4) (2), (4) $\Rightarrow y = -1,\ v = -1$ (1), (3) $\Rightarrow x = \frac{11}{4},\ u = \frac{9}{4}$ $z = \frac{11}{4} - \mathrm{i}$ $w = \frac{9}{4} - \mathrm{i}$
	Forms two of $x + u = 5$ $y - v = 0$ $3x - u = 6$ OE $-3y - v = 4$ or eliminates one complex unknown	1.1a	M1	
	Obtains at least two correct values of x, y, u or v or obtains one of $z^* = \frac{11}{4} + \mathrm{i}$, $w^* = \frac{9}{4} + \mathrm{i}$	1.1a	M1	
	Obtains the values $\frac{11}{4}, -1, \frac{9}{4}, -1$	1.1b	A1	
	Obtains $z = \frac{11}{4} - \mathrm{i}$ $w = \frac{9}{4} - \mathrm{i}$	1.1b	A1	
	Question total		**5**	

Q	Marking instructions	AO	Marks	Typical solution
8	Forms an equation by eliminating y or x	1.1a	M1	$x^2 + \frac{(mx+4)^2}{9} = 1$
	Obtains a correct three-term quadratic equation. PI correct discriminant	1.1b	A1	$9x^2 + m^2x^2 + 8mx + 16 = 9$ $(9 + m^2)x^2 + 8mx + 7 = 0$
	Forms a quadratic equation in m or m^2 using $b^2 - 4ac = 0$	2.2a	M1	For a tangent, $b^2 - 4ac = 0$ $64m^2 - 28(9 + m^2) = 0$ $36m^2 = 252$
	Completes fully correct working to obtain $m = \pm\sqrt{7}$ **AG**	2.1	R1	$m^2 = 7$ $m = \pm\sqrt{7}$
	Question total		**4**	

Q	Marking instructions	AO	Marks	Typical solution
9(a)	Writes down $\frac{e^p - e^{-p}}{2}$ OE	1.2	B1	$\sinh p = \frac{e^p - e^{-p}}{2}$ $= \frac{e^{\ln\left(r+\sqrt{r^2+1}\right)} - e^{-\ln\left(r+\sqrt{r^2+1}\right)}}{2}$ $= \frac{1}{2}\left(r+\sqrt{r^2+1} - \frac{1}{r+\sqrt{r^2+1}}\right)$ $= \frac{1}{2}\left(\frac{\left(r+\sqrt{r^2+1}\right)^2 - 1}{r+\sqrt{r^2+1}}\right)$ $= \frac{1}{2}\left(\frac{\left(r^2+2r\sqrt{r^2+1}+r^2+1\right)-1}{r+\sqrt{r^2+1}}\right)$ $= \frac{1}{2}\left(\frac{2r^2+2r\sqrt{r^2+1}}{r+\sqrt{r^2+1}}\right)$ $= \frac{1}{2}\left(\frac{2r\left(r+\sqrt{r^2+1}\right)}{r+\sqrt{r^2+1}}\right)$ $= r$
	Substitutes p with the given expression and removes e and ln to obtain $r+\sqrt{r^2+1}$ or $\frac{1}{r+\sqrt{r^2+1}}$	3.1a	M1	
	Collects over common denominator $r+\sqrt{r^2+1}$ or Multiplies $\frac{1}{r+\sqrt{r^2+1}}$ by $\frac{r-\sqrt{r^2+1}}{r-\sqrt{r^2+1}}$	1.1a	M1	
	Completes reasoned argument to obtain $\sinh p = r$ AG	2.1	R1	
	Subtotal		**4**	

Q	Marking instructions	AO	Marks	Typical solution
9(b)	Uses $\cosh^2 x = 1 + \sinh^2 x$ to obtain $1 + \sinh^2 x = 2\sinh x + 16$ or Substitutes in correct exponential form to obtain $\left(\frac{e^x + e^{-x}}{2}\right)^2 = 2\left(\frac{e^x - e^{-x}}{2}\right) + 16$	3.1a	M1	Let $s = \sinh x$ $s^2 + 1 = 2s + 16$ $s^2 - 2s - 15 = 0$ $s = -3, 5$ $x = \ln(-3 + \sqrt{10})$, $x = \ln(5 + \sqrt{26})$
	Obtains $(\sinh x =) -3, 5$ or Obtains $(e^x =) 10.099\ldots, 0.162\ldots$	1.1b	A1	
	Uses $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ FT their –3 or 5	1.1a	M1	
	Obtains $\ln(-3 + \sqrt{10})$, $\ln(5 + \sqrt{26})$ and no other solutions	1.1b	A1	
	Subtotal		**4**	

	Question total		**8**	

Q	Marking instructions	AO	Marks	Typical solution
10	Forms the product zw	1.1a	M1	$zw = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}\right)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$
	Obtains $\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} + \mathrm{i}\left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right)$	1.1b	A1	$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} + \mathrm{i}\left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right)$
	States or uses $arg(zw) = arg(z) + arg(w)$	3.1a	M1	Also
	States $\frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$	1.1b	B1	$zw = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$ $= \cos\frac{5\pi}{12} + \mathrm{i}\sin\frac{5\pi}{12}$
	Uses their $zw = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} + \mathrm{i}\left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right)$ to deduce an expression for $\tan\frac{5\pi}{12}$	2.2a	M1	$\tan\frac{5\pi}{12} = \frac{\sin\frac{5\pi}{12}}{\cos\frac{5\pi}{12}} = \frac{\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}}{\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}}$
	Completes a reasoned argument to obtain $\tan\frac{5\pi}{12} = 2 + \sqrt{3}$ AG	2.1	R1	$= 2 + \sqrt{3}$
	Question total		**6**	

Q	Marking instructions	AO	Marks	Typical solution
11(a)	Obtains $2x\tan^{-1}x+\frac{x^2}{1+x^2}$	1.1b	B1	$\frac{d}{dx}(x^2\tan^{-1}x)=2x\tan^{-1}x+\frac{x^2}{1+x^2}$
	Subtotal		**1**	

Q	Marking instructions	AO	Marks	Typical solution
11(b)	Uses result of part **(a)** to obtain an equation involving the required integral.	2.2a	M1	$\frac{d}{dx}(x^2\tan^{-1}x)=2x\tan^{-1}x+\frac{x^2}{1+x^2}$ $\therefore \int 2x\tan^{-1}x\,dx+\int\frac{x^2}{1+x^2}\,dx=x^2\tan^{-1}x$
	Deduces that $\frac{x^2}{1+x^2}=1-\frac{1}{1+x^2}$ or uses the substitution $x=\tan u$ to obtain $\sec^2 u-1$	2.2a	M1	$\int\frac{x^2}{1+x^2}\,dx=\int\frac{1+x^2-1}{1+x^2}\,dx$ $=\int 1\,dx-\int\frac{1}{1+x^2}\,dx$ $=x-\tan^{-1}x+c$
	Obtains $x-\tan^{-1}x$	1.1b	A1	
	Completes a reasoned argument starting with the result from part **(a)** to obtain $x^2\tan^{-1}x-x+\tan^{-1}x+c$	2.1	R1	$\int 2x\tan^{-1}x\,dx=x^2\tan^{-1}x-\int\frac{x^2}{1+x^2}\,dx$ $=x^2\tan^{-1}x-x+\tan^{-1}x+c$
	Subtotal		**4**	

	Question total		**5**	

Q	Marking instructions	AO	Marks	Typical solution
12(a)	Forms the product of the matrix and the direction/position vector of L_1	2.2a	M1	$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 6 \\ -5 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix}$ = direction vector of L_2 $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix} = 28$ $\cos\theta = \frac{28}{\sqrt{11}\sqrt{122}}$ $\theta = 0.701$ (3dp)
	Obtains the correct direction vector of L_2 possibly embedded	1.1b	A1	
	Use the scalar product of the direction vectors of L_1 and their L_2 to obtain their correct 28.	1.1a	M1	
	Completes a reasoned argument to obtain 0.701 Must see $\cos\theta = \frac{28}{\sqrt{11}\sqrt{122}}$ or $\cos\theta = 0.7643\ldots$ or $\theta = 0.7007...$	2.1	R1	
	Subtotal		**4**	

Q	Marking instructions	AO	Marks	Typical solution
12(b)	Forms the product of the matrix and a point on L_1	2.2a	M1	$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 6 \\ -5 & 2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 26 \\ -19 \end{bmatrix}$
	Obtains a correct point on $\begin{bmatrix} 10 \\ 26 \\ -19 \end{bmatrix} + \mu\begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix}$	1.1b	A1	Vector connecting L_1 to L_2
	Obtains a vector connecting a point on each line. or Obtains the equations of two correct parallel planes PI	1.1b	B1	$\overrightarrow{AB} = \begin{bmatrix} 10 \\ 26 \\ -19 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \\ -20 \end{bmatrix}$
	Obtains a vector perpendicular to both lines. or Calculates the scalar product of the general vector between the lines with both direction vectors to obtain a pair of simultaneous equations.	3.1a	M1	Vector perpendicular to both lines $\mathbf{n} = \begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -21 \\ 9 \\ 6 \end{bmatrix}$
	Uses a correct method to obtain the shortest distance between the lines. Condone a negative distance or Obtains solutions to their pair of simultaneous equations and obtains the vector between the two closest points	3.1a	M1	$\mathbf{n} \bullet \overrightarrow{AB} = -30$ Distance $= \left\| \dfrac{\mathbf{n} \bullet \overrightarrow{AB}}{\|\mathbf{n}\|} \right\| = \dfrac{10}{\sqrt{62}} = \dfrac{5\sqrt{62}}{31}$
	Obtains $\dfrac{5\sqrt{62}}{31}$ OE	1.1b	A1	
	Subtotal		**6**	

	Question total		**10**	

Alternative typical solutions

Alternative Typical Solution 1	Alternative Typical Solution 2
$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 6 \\ -5 & 2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 26 \\ -19 \end{bmatrix}$ Equation of $L_2 : \mathbf{r} = \begin{bmatrix} 10 \\ 26 \\ -19 \end{bmatrix} + \mu\begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix}$ General vector $\overrightarrow{AB} = \begin{bmatrix} 10+5\mu \\ 26+9\mu \\ -19+4\mu \end{bmatrix} - \begin{bmatrix} 4+\lambda \\ 2+3\lambda \\ 1-\lambda \end{bmatrix} = \begin{bmatrix} 6+5\mu-\lambda \\ 24+9\mu-3\lambda \\ -20+4\mu+\lambda \end{bmatrix}$ Scalar product $0 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 6+5\mu-\lambda \\ 24+9\mu-3\lambda \\ -20+4\mu+\lambda \end{bmatrix} = 98+28\mu-11\lambda$ $11\lambda - 28\mu = 98$ $0 = \begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 6+5\mu-\lambda \\ 24+9\mu-3\lambda \\ -20+4\mu+\lambda \end{bmatrix} = 166+122\mu-28\lambda$ $28\lambda - 122\mu = 166$ $\lambda = \frac{406}{31}$, $\mu = \frac{51}{31}$ $\overrightarrow{AB} = \begin{bmatrix} 35/31 \\ -15/31 \\ -10/31 \end{bmatrix}$ Distance = $\frac{5\sqrt{62}}{31}$	$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 6 \\ -5 & 2 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 26 \\ -19 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 9 \\ 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -21 \\ 9 \\ 6 \end{bmatrix}$ Parallel planes $\Pi_1 \quad 21x - 9y - 6z = 21\times4 - 9\times2 - 6\times1 = 60$ $\Pi_2 \quad 21x - 9y - 6z = 21\times10 - 9\times26 - 6\times-19 = 90$ distance $= \frac{90-60}{\sqrt{21^2+9^2+6^2}} = \frac{5\sqrt{62}}{31}$

Q	Marking instructions	AO	Marks	Typical solution
13(a)	Expands $(\cos\theta + \mathrm{i}\sin\theta)^3$ PI correct real part	1.1a	M1	$\cos 3\theta + \mathrm{i}\sin 3\theta = (\cos\theta + \mathrm{i}\sin\theta)^3$ $= \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$ Equating real parts $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ $= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$ $= 4\cos^3\theta - 3\cos\theta$
	Equates real parts and uses $\sin^2\theta = 1 - \cos^2\theta$	1.1a	M1	
	Completes a reasoned argument using de Moivre's theorem to show $\cos 3\theta$ $= 4\cos^3\theta - 3\cos\theta$ AG	2.1	R1	
	Subtotal		**3**	

Q	Marking instructions	AO	Marks	Typical solution
13(b)	Equates imaginary parts and uses $\cos^2\theta = 1 - \sin^2\theta$	1.1a	M1	Equating imaginary parts $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ $= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$ $= 3\sin\theta - 4\sin^3\theta$
	Obtains $3\sin\theta - 4\sin^3\theta$	1.1b	A1	
	Subtotal		**2**	

Q	Marking instructions	AO	Marks	Typical solution
13(c)	Substitutes expressions for $\cos 3\theta$ and their $\sin 3\theta$ into $\cot 3\theta = \dfrac{\cos 3\theta}{\sin 3\theta}$ or $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta}$	1.1b	B1F	$\cot 3\theta = \dfrac{\cos 3\theta}{\sin 3\theta} = \dfrac{4\cos^3\theta - 3\cos\theta}{3\sin\theta - 4\sin^3\theta}$
	Manipulates their rational function of $\sin\theta$ and $\cos\theta$ to obtain at least one instance of $\cot\theta$ or $\tan\theta$	3.1a	M1	$= \dfrac{4\left(\cos^3\theta/\sin^3\theta\right) - 3\left(\cos\theta/\sin^3\theta\right)}{3\left(\sin\theta/\sin^3\theta\right) - 4\left(\sin^3\theta/\sin^3\theta\right)}$
	Manipulates their rational function of $\sin\theta$ and $\cos\theta$ to obtain only $\cot\theta$ (and $\operatorname{cosec}\theta$) terms	3.1a	M1	$= \dfrac{4\cot^3\theta - 3\cot\theta\operatorname{cosec}^2\theta}{3\operatorname{cosec}^2\theta - 4}$ $= \dfrac{4\cot^3\theta - 3\cot\theta\left(1+\cot^2\theta\right)}{3\left(1+\cot^2\theta\right) - 4}$
	Completes a reasoned argument from the final or intermediate results in parts **(a)** and **(b)** to show $\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$ AG	2.1	R1	$\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$
	Subtotal		**4**	

	Question total		**9**	

Q	Marking instructions	AO	Marks	Typical solution
14	Writes $e^{\int \tanh x \mathrm{d}x}$	3.1a	M1	Integrating factor = $e^{\int \tanh x \mathrm{d}x}$ $\int \tanh x \mathrm{d}x = \ln(\cosh x)$ Integrating factor = $\cosh x$ $y\cosh x = \int \sinh^3 x \cosh x \,\mathrm{d}x$ $= \frac{1}{4}\sinh^4 x + c$ When $x = \ln 2$, $\sinh x = \frac{3}{4}$ and $\cosh x = \frac{5}{4}$ $3 \times \frac{5}{4} = \frac{1}{4} \times \frac{81}{256} + c$ $c = \frac{3759}{1024}$ $y\cosh x = \frac{1}{4}\sinh^4 x + \frac{3759}{1024}$
	Obtains $\cosh x$	1.1b	A1	
	Obtains $y\cosh x$	1.1b	B1	
	Writes down $\int \sinh^3 x \cosh x \,\mathrm{d}x$ (PI)	1.1a	M1	
	Obtains $k\sinh^4 x$ or $k_1\cosh 4x + k_2\cosh 2x$ (Accept equivalent exponential form)	1.1b	A1	
	Substitutes $x = \ln 2$ and $y = 3$ into $y\cosh x = k\sinh^4 x + c$ or $y\cosh x$ $= k_1\cosh 4x + k_2\cosh 2x + c$ (Accept equivalent exponential form) and uses to evaluate c	1.1a	M1	
	Deduces $y\cosh x = \frac{1}{4}\sinh^4 x + \frac{3759}{1024}$ Or $y\cosh x$ $= \frac{1}{32}\cosh 4x - \frac{1}{8}\cosh 2x + \frac{3855}{1024}$ ACF. ISW	2.2a	A1	
	Question total		**7**	

Q	Marking instructions	AO	Marks	Typical solution
15	Obtains $\dot{x} = \frac{9}{2}t^2$ and $\dot{y} = \frac{9}{2}t^{\frac{7}{2}}$	1.1b	B1	$\dot{x} = \frac{9}{2}t^2$ and $\dot{y} = \frac{9}{2}t^{\frac{7}{2}}$ $\dot{x}^2 + \dot{y}^2 = \frac{81}{4}(t^4 + t^7)$ Arc length $s = \int_0^2 \sqrt{\frac{81}{4}(t^4 + t^7)}\,dt$ $= \frac{9}{2}\int_0^2 t^2\sqrt{(1+t^3)}\,dt$ $= \left[(1+t^3)^{\frac{3}{2}}\right]_0^2$ $= 9^{\frac{3}{2}} - 1^{\frac{3}{2}}$ $= 26$
	Obtains their $\sqrt{\dot{x}^2 + \dot{y}^2}$	1.1a	M1	
	Writes integrand in the form $k\,t^2\sqrt{(1+t^3)}$	3.1a	M1	
	Obtains $\lambda\,(1+t^3)^{\frac{3}{2}}$	2.2a	A1	
	Completes a reasoned argument to obtain 26 AG	2.1	R1	
	Question total		**5**	

Q	Marking instructions	AO	Marks	Typical solution
16(a)	Obtains $2+\tan\frac{\pi}{4}$	2.2a	M1	At A, $r = 2 + \tan\frac{\pi}{4} = 3$
	Completes a reasoned argument to obtain $\text{Area} = 3\sqrt{2}$ AG	2.1	R1	Area of triangle OAB $= \frac{1}{2} \times 4 \times 3 \times \sin\frac{\pi}{4}$ $= 3\sqrt{2}$
	Subtotal		**2**	

Q	Marking instructions	AO	Marks	Typical solution
16(b)	Writes down $\frac{1}{2}\int(2+\tan\theta)^2 \mathrm{d}\theta$	1.1a	M1	Area enclosed by curve $=\frac{1}{2}\int_0^{\frac{\pi}{4}}(2+\tan\theta)^2\mathrm{d}\theta$ $=\frac{1}{2}\int_0^{\frac{\pi}{4}}(4+4\tan\theta+\tan^2\theta)\mathrm{d}\theta$ $=\frac{1}{2}\int_0^{\frac{\pi}{4}}(3+4\tan\theta+\sec^2\theta)\mathrm{d}\theta$ $=\frac{1}{2}\left[3\theta+4\ln(\sec\theta)+\tan\theta\right]_0^{\frac{\pi}{4}}$ $=\frac{1}{2}\left(\frac{3\pi}{4}+4\ln(\sqrt{2})+1-0\right)$ $=\frac{3\pi}{8}+\ln 2+\frac{1}{2}$ Shaded area $=3\sqrt{2}-\frac{3\pi}{8}-\ln 2-\frac{1}{2}$
	Obtains a correct integral for polar area. $\frac{1}{2}\int(4+4\tan\theta+\tan^2\theta)\,\mathrm{d}\theta$	1.1b	A1	
	Replaces $\tan^2\theta$ in the integrand with $\sec^2\theta-1$ Condone $\pm\sec^2\theta\pm1$	3.1a	M1	
	Obtains $3\theta+4\ln(\sec\theta)+\tan\theta$	1.1b	A1	
	Obtains $k\left(\frac{3\pi}{4}+2\ln 2+1\right)$ OE	1.1b	A1	
	Obtains a numerical value for the area of the shaded region using $3\sqrt{2}-$ their area of unshaded region	2.4	M1	
	Completes a reasoned argument to obtain $=3\sqrt{2}-\frac{3\pi}{8}-\ln 2-\frac{1}{2}$	2.1	R1	
	Subtotal		**7**	

	Question total		**9**	

Q	Marking instructions	AO	Marks	Typical solution
17	Completes the square to obtain $(x+a)^2+b$	3.1a	M1	$x^2+6x+8=(x+3)^2-1$ Let $x+3=\cosh\theta$ Then $\dfrac{dx}{d\theta}=\sinh\theta$ Let $I=\int_{-2}^{1}\sqrt{x^2+6x+8}\,dx$ When $x=-2$, $\cosh\theta=1$ and $\theta=0$ When $x=1$, $\cosh\theta=4$ and $\theta=\cosh^{-1}(4)$ $I=\int_0^{\cosh^{-1}(4)}\sqrt{\sinh^2\theta}\,\sinh\theta\,d\theta$ $=\int_0^{\cosh^{-1}(4)}\sinh^2\theta\,d\theta$ $I=\frac{1}{2}\int_0^{\cosh^{-1}(4)}(\cosh2\theta-1)d\theta$ $=\frac{1}{2}\left[\frac{\sinh2\theta}{2}-\theta\right]_0^{\cosh^{-1}(4)}$ $=\frac{1}{2}[\sinh\theta\cosh\theta-\theta]_0^{\cosh^{-1}(4)}$ When $\cosh\theta=4$, $\sinh\theta=\sqrt{4^2-1}=\sqrt{15}$ $I=\frac{1}{2}\left(4\sqrt{15}-\cosh^{-1}(4)\right)-0$ $=2\sqrt{15}-\frac{1}{2}\cosh^{-1}(4)$
	Writes $x+a=k\cosh\theta$ or $x+a=k\sinh\theta$	2.2a	M1	
	Obtains $\int\sinh^2\theta\,d\theta$	1.1b	A1	
	Uses $\sinh^2\theta=\frac{1}{2}\cosh2\theta-\frac{1}{2}$ Condone $\sinh^2\theta=\pm\frac{1}{2}\cosh2\theta\pm\frac{1}{2}$	3.1a	M1	
	Obtains $\frac{1}{2}\left[\frac{\sinh2\theta}{2}-\theta\right]$ OE	1.1b	A1	
	Uses $\cosh^2\theta-\sinh^2\theta=1$ when $x=1$ to obtain $\sinh\theta=\sqrt{15}$	2.2a	B1	
	Completes a reasoned argument to obtain $2\sqrt{15}-\frac{1}{2}\cosh^{-1}(4)$ AG	2.1	R1	
	Question total		**7**	

Q	Marking instructions	AO	Marks	Typical solution
18(a)	Forms equilibrium force equation with three correct terms. Condone sign errors.	3.1b	M1	$7e_A = 3e_C + 0.5g$ $3 = e_A + e_C$ $e_A = 1.39,\ e_C = 1.61$
	Uses the equation $3 = e_A + e_C$ and their force equation to obtain the extensions.	3.1b	M1	
	Obtains 1.39 and 1.61	1.1b	A1	
	Subtotal		**3**	

Q	Marking instructions	AO	Marks	Typical solution
18(b)(i)	Forms at least one correct expression in x for the tension ie $7(e_A + x)$ or $3(e_C - x)$ FT their e_A or e_C	3.1b	B1F	$0.5\ddot{x} = 0.5g + 3(e_C - x) - 7(e_A + x) - 4.5\dot{x}$ $0.5\ddot{x} = 0.5g + 3(1.61 - x) - 7(1.39 + x) - 4.5\dot{x}$ $0.5\ddot{x} = 4.9 + 4.83 - 3x - 9.73 - 7x - 4.5\dot{x}$ $\ddot{x} + 9\dot{x} + 20x = 0$
	Forms an equation of motion with five terms (with at least two terms correct). Accept "a" for $\ddot{x}$ And "v" for $\dot{x}$ Condone sign errors on the terms.	3.1b	M1	
	Completes a reasoned argument to obtain $\ddot{x} + 9\dot{x} + 20x = 0$ OE AG	2.1	R1	
	Subtotal		**3**	

Q	Marking instructions	AO	Marks	Typical solution
18(b)(ii)	Obtains solution from their three term Auxiliary Equation.	3.1a	M1	$0 = \lambda^2 + 9\lambda + 20$ $\lambda = -4$ or $\lambda = -5$ $x = Ae^{-4t} + Be^{-5t}$ $\dot{x} = -4Ae^{-4t} - 5Be^{-5t}$ $x = 0.6$, $\dot{x} = 0$, $t = 0$ $0.6 = A + B$ $0 = -4A - 5B$ $\Rightarrow A = 3$ and $B = -2.4$ $x = 3e^{-4t} - 2.4e^{-5t}$
	Obtains $Ae^{-4t} + Be^{-5t}$	1.1b	A1	
	Uses $x = 0.6$ when $t = 0$ to find an equation in two variables	3.4	M1	
	Sets their correct $\dot{x} = 0$ when $t = 0$ to find an equation in two variables	3.3	M1	
	Obtains $x = 3e^{-4t} - 2.4e^{-5t}$ OE	1.1b	A1	
	Subtotal		**5**	

Q	Marking instructions	AO	Marks	Typical solution
18(c)	Gives a valid limitation with reference to the size of the tube.	3.5b	E1	The resistance force in a thin tube might not be the same as in a large bath.
	Subtotal		**1**	

	Question total		**12**	

	Question paper total		**100**	