A Level
Further Mathematics

Sample Assessment Materials

Pearson Edexcel Level 3 Advanced GCE in Further Mathematics (9FM0)

First teaching from September 2017

First certification from 2019

Issue 1

Edexcel, BTEC and LCCI qualifications

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Contents

Introduction

The Pearson Edexcel Level 3 Advanced GCE in Further Mathematics is designed for use in schools and colleges. It is part of a suite of AS/A Level qualifications offered by Pearson.

These sample assessment materials have been developed to support this qualification and will be used as the benchmark to develop the assessment students will take.

The booklet '*Mathematical Formulae and Statistical Tables*' will be provided for use with these assessments and can be downloaded from our website, qualifications.pearson.com.

General marking guidance

- All candidates must receive the same treatment. Examiners must mark the last candidate in exactly the same way as they mark the first.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than be penalised for omissions.
- Examiners should mark according to the mark scheme – not according to their perception of where the grade boundaries may lie.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate’s response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification/indicative content will not be exhaustive. However different examples of responses will be provided at standardisation.
- When examiners are in doubt regarding the application of the mark scheme to a candidate’s response, a senior examiner must be consulted before a mark is given.
- Crossed-out work should be marked **unless** the candidate has replaced it with an alternative response.

Specific guidance for mathematics

1. These mark schemes use the following types of marks:
    - **M** marks: Method marks are awarded for ‘knowing a method and attempting to apply it’, unless otherwise indicated.
    - **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
    - **B** marks are unconditional accuracy marks (independent of M marks)
    - Marks should not be subdivided.
2. Abbreviations

    These are some of the traditional marking abbreviations that may appear in the mark schemes.

bod	benefit of doubt
ft	follow through
√	this symbol is used for correct ft
cao	correct answer only
cso	correct solution only. There must be no errors in this part of the question to obtain this mark
isw	ignore subsequent working
awrt	answers which round to
SC:	special case
o.e.	or equivalent (and appropriate)
d… or **dep**	dependent
indep	independent
dp	decimal places
sf	significant figures
*****	The answer is printed on the paper or ag- answer given

- [or d… The second mark is dependent on gaining the first mark

3. All M marks are follow through.

   All A marks are ‘correct answer only’ (cao.), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but answers that don’t logically make sense e.g. if an answer given for a probability is >1 or <0, should never be awarded A marks.

4. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

5. Where a candidate has made multiple responses and indicates which response they wish to submit, examiners should mark this response. If there are several attempts at a question which have not been crossed out, examiners should mark the final answer which is the answer that is the most complete.

6. Ignore wrong working or incorrect statements following a correct answer.

7. Mark schemes will firstly show the solution judged to be the most common response expected from candidates. Where appropriate, alternative answers are provided in the notes. If examiners are not sure if an answer is acceptable, they will check the mark scheme to see if an alternative answer is given for the method used. If no such alternative answer is provided but deemed to be valid, examiners must escalate the response to a senior examiner to review.

Write your name here

Surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number ☐☐☐☐☐

Candidate Number ☐☐☐☐

Further Mathematics

Advanced
Paper 1: Core Pure Mathematics 1

Sample Assessment Material for first teaching September 2017
Time: 1 hour 30 minutes

Paper Reference
9FM0/01

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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Pearson

Answer ALL questions. Write your answers in the spaces provided.

1. Prove that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(an+b)}{12(n+2)(n+3)}$$

where a and b are constants to be found.

(5)

Question 1 continued

(Total for Question 1 is 5 marks)

2. Prove by induction that for all positive integers n,

$$f(n) = 2^{3n+1} + 3(5^{2n+1})$$

is divisible by 17

(6)

Question 2 continued

(Total for Question 2 is 6 marks)

3. $$f(z) = z^4 + az^3 + 6z^2 + bz + 65$$

where a and b are real constants.

Given that $z = 3 + 2\mathbf{i}$ is a root of the equation $f(z) = 0$, show the roots of $f(z) = 0$ on a single Argand diagram.

(9)

Question 3 continued

(Total for Question 3 is 9 marks)

4.

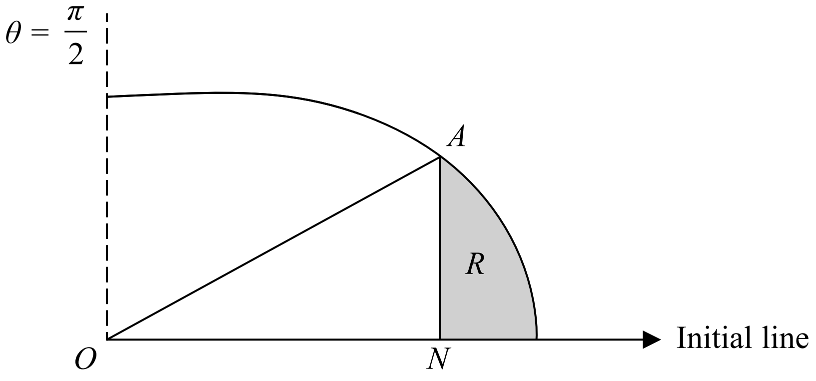


Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 4 + \cos 2\theta \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point A on C, the value of r is $\frac{9}{2}$

The point N lies on the initial line and AN is perpendicular to the initial line.

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the initial line and the line AN.

Find the exact area of the shaded region R, giving your answer in the form $p\pi + q\sqrt{3}$ where p and q are rational numbers to be found.

(9)

Question 4 continued

(Total for Question 4 is 9 marks)

5. A pond initially contains 1000 litres of unpolluted water.

The pond is leaking at a constant rate of 20 litres per day.

It is suspected that contaminated water flows into the pond at a constant rate of 25 litres per day and that the contaminated water contains 2 grams of pollutant in every litre of water.

It is assumed that the pollutant instantly dissolves throughout the pond upon entry.

Given that there are x grams of the pollutant in the pond after t days,

(a) show that the situation can be modelled by the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 50 - \frac{4x}{200 + t}$$

(4)

(b) Hence find the number of grams of pollutant in the pond after 8 days.

(5)

(c) Explain how the model could be refined.

(1)

Question 5 continued

(Total for Question 5 is 10 marks)

6.

$$f(x) = \frac{x + 2}{x^2 + 9}$$

(a) Show that

$$\int f(x)\,dx = A\ln(x^2 + 9) + B\arctan\left(\frac{x}{3}\right) + c$$

where c is an arbitrary constant and A and B are constants to be found.

(4)

(b) Hence show that the mean value of $f(x)$ over the interval $[0, 3]$ is

$$\frac{1}{6}\ln 2 + \frac{1}{18}\pi$$

(3)

(c) Use the answer to part (b) to find the mean value, over the interval $[0, 3]$, of

$$f(x) + \ln k$$

where k is a positive constant, giving your answer in the form $p + \frac{1}{6}\ln q$, where p and q are constants and q is in terms of k.

(2)

Question 6 continued

(Total for Question 6 is 9 marks)

7.

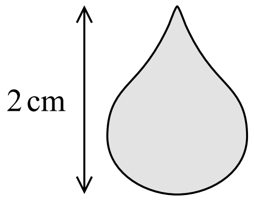


Figure 2

Figure 2 shows the image of a gold pendant which has height 2 cm. The pendant is modelled by a solid of revolution of a curve C about the y-axis. The curve C has parametric equations

$$x = \cos\theta + \frac{1}{2}\sin 2\theta, \quad y = -(1 + \sin\theta) \quad 0 \leqslant \theta \leqslant 2\pi$$

(a) Show that a Cartesian equation of the curve C is

$$x^2 = -(y^4 + 2y^3)$$

(4)

(b) Hence, using the model, find, in cm^3, the volume of the pendant.

(4)

Question 7 continued

Question 7 continued

Question 7 continued

(Total for Question 7 is 8 marks)

8. The line l_1 has equation $\dfrac{x-2}{4} = \dfrac{y-4}{-2} = \dfrac{z+6}{1}$

The plane Π has equation $x - 2y + z = 6$

The line l_2 is the reflection of the line l_1 in the plane Π.

Find a vector equation of the line l_2

(7)

Question 8 continued

Question 8 continued

Question 8 continued

(Total for Question 8 is 7 marks)

9. A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, x metres, of the top of the capsule below its initial position at time t seconds is modelled by the differential equation,

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + x = 200\cos t, \qquad t \geqslant 0$$

where m is the mass of the capsule including its passengers, in thousands of kilograms.

The maximum permissible weight for the capsule, including its passengers, is 30 000 N.

Taking the value of g to be $10\,\mathrm{ms}^{-2}$ and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of m is 3

(ii) show that a particular solution to the differential equation is

$$x = 40\sin t - 20\cos t$$

(iii) hence find the general solution of the differential equation.

(8)

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

(4)

Question 9 continued

Question 9 continued

Question 9 continued

(Total for Question 9 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS

Paper 1: Core Pure Mathematics 1 Mark Scheme

Question	Scheme	Marks	AOs
1	$\frac{1}{(r+1)(r+3)} \equiv \frac{A}{(r+1)} + \frac{B}{(r+3)} \Rightarrow A = ...,\ B = ...$	M1	3.1a
	$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} =$ $\frac{1}{2\times 2} - \frac{1}{2\times 4} + \frac{1}{2\times 3} - \frac{1}{2\times 5} + ... + \frac{1}{2n} - \frac{1}{2(n+2)} + \frac{1}{2(n+1)} - \frac{1}{2(n+3)}$	M1	2.1
	$= \frac{1}{4} + \frac{1}{6} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$	A1	2.2a
	$= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{12(n+2)(n+3)}$	M1	1.1b
	$= \frac{n(5n+13)}{12(n+2)(n+3)}$	A1	1.1b
		(5)	
	Alternative by induction: $n = 1 \Rightarrow \frac{1}{8} = \frac{a+b}{12\times 3\times 4},\ n = 2 \Rightarrow \frac{1}{8} + \frac{1}{15} = \frac{2(2a+b)}{12\times 4\times 5}$ $a + b = 18,\ 2a + b = 23 \Rightarrow a = ...,\ b = ...$	M1	3.1a
	Assume true for $n = k$ so $\sum_{r=1}^{k} \frac{1}{(r+1)(r+3)} = \frac{k(5k+13)}{12(k+2)(k+3)}$		
	$\sum_{r=1}^{k+1} \frac{1}{(r+1)(r+3)} = \frac{k(5k+13)}{12(k+2)(k+3)} + \frac{1}{(k+2)(k+4)}$	M1	2.1
	$\frac{k(5k+13)}{12(k+2)(k+3)} + \frac{1}{(k+2)(k+4)} = \frac{k(5k+13)(k+4) + 12(k+3)}{12(k+2)(k+3)(k+4)}$	A1	2.2a
	$= \frac{5k^3 + 33k^2 + 52k + 12k + 36}{12(k+2)(k+3)(k+4)} = \frac{(k+1)(k+2)(5k+18)}{12(k+2)(k+3)(k+4)}$	M1	1.1b
	$= \frac{(\underline{k+1})(5(\underline{k+1})+13)}{12(\underline{k+1}+2)(\underline{k+1}+3)}$ So true for $n = k + 1$ So $\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(5n+13)}{12(n+2)(n+3)}$	A1	1.1b
		(5)	
			(5 marks)

Question 1 notes:

Main Scheme

M1: Valid attempt at partial fractions
M1: Starts the process of differences to identify the relevant fractions at the start and end
A1: Correct fractions that do not cancel
M1: Attempt common denominator
A1: Correct answer

Alternative by Induction:

M1: Uses $n = 1$ and $n = 2$ to identify values for a and b
M1: Starts the induction process by adding the $(k + 1)^{\text{th}}$ term to the sum of k terms
A1: Correct single fraction
M1: Attempt to factorise the numerator
A1: Correct answer and conclusion

Question	Scheme	Marks	AOs
2	When $n = 1$, $2^{3n+1}+3\left(5^{2n+1}\right)=16+375=391$ 391 = 17 × 23 so the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $2^{3k+1}+3\left(5^{2k+1}\right)$ is divisible by 17	M1	2.4
	$\mathrm{f}\left(k+1\right)-\mathrm{f}\left(k\right)=2^{3k+4}+3\left(5^{2k+3}\right)-2^{3k+1}-3\left(5^{2k+1}\right)$	M1	2.1
	$=7\times2^{3k+1}+7\times3\left(5^{2k+1}\right)+17\times3\left(5^{2k+1}\right)$		
	$=7\mathrm{f}\left(k\right)+17\times3\left(5^{2k+1}\right)$	A1	1.1b
	$\mathrm{f}\left(k+1\right)=8\mathrm{f}\left(k\right)+17\times3\left(5^{2k+1}\right)$	A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n	A1	2.4
		(6)	
			(6 marks)

Notes:

B1: Shows the statement is true for $n = 1$

M1: Assumes the statement is true for $n = k$

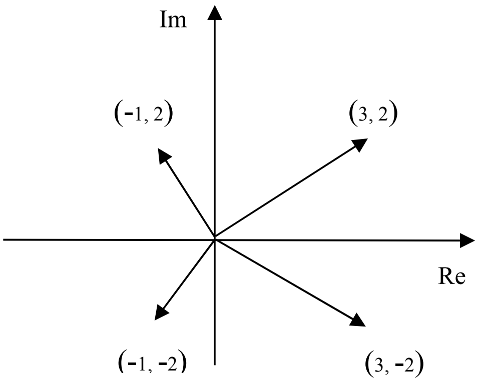
M1: Attempts f(k+1) – f(k)

A1: Correct expression in terms of f(k)

A1: Correct expression in terms of f(k)

A1: Obtains a correct expression for f(k + 1)

A1: Correct complete conclusion

Question	Scheme	Marks	AOs
3	$z = 3 - 2\text{i}$ is also a root	B1	1.2
	$(z-(3+2\text{i}))(z-(3-2\text{i})) = \ldots$ **or** Sum of roots = 6, Product of roots = 13 $\Rightarrow \ldots$	M1	3.1a
	$= z^2 - 6z + 13$	A1	1.1b
	$(z^4 + az^3 + 6z^2 + bz + 65) = (z^2 - 6z + 13)(z^2 + cz + 5) \Rightarrow c = \ldots$	M1	3.1a
	$z^2 + 2z + 5 = 0$	A1	1.1b
	$z^2 + 2z + 5 = 0 \Rightarrow z = \ldots$	M1	1.1a
	$z = -1 \pm 2\text{i}$	A1	1.1b
		B1 3 ± 2i Plotted correctly	1.1b
		B1ft −1 ± 2i Plotted correctly	1.1b
			(9 marks)

Notes:

B1: Identifies the complex conjugate as another root

M1: Uses the conjugate pair and a correct method to find a quadratic factor

A1: Correct quadratic

M1: Uses the given quartic and their quadratic to identify the value of *c*

A1: Correct 3TQ

M1: Solves their second quadratic

A1: Correct second conjugate pair

B1: First conjugate pair plotted correctly and labelled

B1ft: Second conjugate pair plotted correctly and labelled (Follow through their second conjugate pair)

Question	Scheme	Marks	AOs
4	$4+\cos 2\theta=\frac{9}{2}\Rightarrow\theta=...$	M1	3.1a
	$\theta=\frac{\pi}{6}$	A1	1.1b
	$\frac{1}{2}\int\left(4+\cos 2\theta\right)^2\text{d}\theta=\frac{1}{2}\int\left(16+8\cos 2\theta+\cos^2 2\theta\right)\text{d}\theta$	M1	3.1a
	$\cos^2 2\theta=\frac{1}{2}+\frac{1}{2}\cos 4\theta\Rightarrow A=\frac{1}{2}\int\left(16+8\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)\text{d}\theta$	M1	3.1a
	$=\frac{1}{2}\left[16\theta+4\sin 2\theta+\frac{\sin 4\theta}{8}+\frac{\theta}{2}\right]$	A1	1.1b
	Using limits 0 and their $\frac{\pi}{6}$: $\frac{1}{2}\left[\frac{33\pi}{12}+2\sqrt{3}+\frac{\sqrt{3}}{16}-(0)\right]$	M1	1.1b
	Area of triangle = $\frac{1}{2}\left(r\cos\theta\right)\left(r\sin\theta\right)=\frac{1}{2}\times\frac{81}{4}\times\frac{1}{2}\times\frac{\sqrt{3}}{2}$	M1	3.1a
	Area of $R=\frac{33\pi}{24}+\frac{33\sqrt{3}}{32}-\frac{81\sqrt{3}}{32}$	M1	1.1b
	$=\frac{11}{8}\pi-\frac{3\sqrt{3}}{2}\left(p=\frac{11}{8},\ q=-\frac{3}{2}\right)$	A1	1.1b
			(9 marks)

Notes:

M1: Realises the angle for *A* is required and attempts to find it
A1: Correct angle
M1: Uses a correct area formula and squares *r* to achieve a 3TQ integrand in cos 2θ
M1: Use of the correct double angle identity on the integrand to achieve a suitable form for integration
A1: Correct integration
M1: Correct use of limits
M1: Identifies the need to subtract the area of a triangle and so finds the area of the triangle
M1: Complete method for the area of *R*
A1: Correct final answer

Question	Scheme	Marks	AOs
5(a)	Pond contains 1000 + 5t litres after t days	M1	3.3
	If x is the amount of pollutant in the pond after t days Rate of pollutant out = $20 \times \frac{x}{1000+5t}$ g per day	M1	3.3
	Rate of pollutant in = 25 × 2 g = 50g per day	B1	2.2a
	$\frac{dx}{dt} = 50 - \frac{4x}{200+t}$ *	A1*	1.1b
		(4)	
(b)	$I = e^{\int \frac{4}{200+t} dt} = (200+t)^4 \Rightarrow x(200+t)^4 = \int 50(200+t)^4 \, dt$	M1	3.1b
	$x(200+t)^4 = 10(200+t)^5 + c$	A1	1.1b
	$x = 0,\ t = 0 \Rightarrow\ c = -3.2 \times 10^{12}$	M1	3.4
	$t = 8 \Rightarrow\ x = 10(200+8) - \frac{3.2 \times 10^{12}}{(200+8)^4}$	M1	1.1b
	= 370g	A1	2.2b
		(5)	
(c)	e.g. • The model should take into account the fact that the pollutant does not dissolve throughout the pond upon entry • The rate of leaking could be made to vary with the volume of water in the pond	B1	3.5c
		(1)	
		(10 marks)	

Notes:

(a)

M1: Forms an expression of the form 1000 + kt for the volume of water in the pond at time t

M1: Expresses the amount of pollutant out in terms of x and t

B1: Correct interpretation for pollutant entering the pond

A1*: Puts all the components together to form the correct differential equation

(b)

M1: Uses the model to find the integrating factor and attempts solution of their differential equation

A1: Correct solution

M1: Interprets the initial conditions to find the constant of integration

M1: Uses their solution to the problem to find the amount of pollutant after 8 days

A1: Correct number of grams

(c)

B1: Suggests a suitable refinement to the model

Question	Scheme	Marks	AOs
6(a)	$\mathrm{f}(x)=\frac{x+2}{x^2+9}=\frac{x}{x^2+9}+\frac{2}{x^2+9}$	B1	3.1a
	$\int\frac{x}{x^2+9}\,\mathrm{d}x=k\ln\left(x^2+9\right)(+c)$	M1	1.1b
	$\int\frac{2}{x^2+9}\,\mathrm{d}x=k\arctan\left(\frac{x}{3}\right)(+c)$	M1	1.1b
	$\int\frac{x+2}{x^2+9}\,\mathrm{d}x=\frac{1}{2}\ln\left(x^2+9\right)+\frac{2}{3}\arctan\left(\frac{x}{3}\right)+c$	A1	1.1b
		(4)	
(b)	$\int_0^3\mathrm{f}(x)\,\mathrm{d}x=\left[\frac{1}{2}\ln\left(x^2+9\right)+\frac{2}{3}\arctan\left(\frac{x}{3}\right)\right]_0^3$ $=\frac{1}{2}\ln 18+\frac{2}{3}\arctan\left(\frac{3}{3}\right)-\left(\frac{1}{2}\ln 9+\frac{2}{3}\arctan(0)\right)$ $=\frac{1}{2}\ln\frac{18}{9}+\frac{2}{3}\arctan\left(\frac{3}{3}\right)$	M1	1.1b
	Mean value = $\frac{1}{3-0}\left(\frac{1}{2}\ln 2+\frac{\pi}{6}\right)$	M1	2.1
	$\frac{1}{6}\ln 2+\frac{1}{18}\pi$ *	A1*	2.2a
		(3)	
(c)	$\frac{1}{6}\ln 2+\frac{1}{18}\pi+\ln k$	M1	2.2a
	$\frac{1}{6}\ln 2k^6+\frac{1}{18}\pi$	A1	1.1b
		(2)	
			(9 marks)

Notes:

(a)

B1: Splits the fraction into two correct separate expressions

M1: Recognises the required form for the first integration

M1: Recognises the required form for the second integration

A1: Both expressions integrated correctly and added together with constant of integration included

(b)

M1: Uses limits correctly and combines logarithmic terms

M1: Correctly applies the method for the mean value for their integration

A1*: Correct work leading to the given answer

(c)

M1: Realises that the effect of the transformation is to increase the mean value by ln *k*

A1: Combines ln's correctly to obtain the correct expression

Question	Scheme	Marks	AOs
7(a)	$x = \cos\theta + \sin\theta\cos\theta = -y\cos\theta$	M1	2.1
	$\sin\theta = -y-1$	M1	2.1
	$\left(\frac{x}{-y}\right)^2 = 1-(-y-1)^2$	M1	2.1
	$x^2 = -\left(y^4+2y^3\right)*$	A1*	1.1b
		(4)	
(b)	$V = \pi\int x^2\,\mathrm{d}y = \pi\int -\left(y^4+2y^3\right)\,\mathrm{d}y$	M1	3.4
	$=\pi\left[-\left(\frac{y^5}{5}+\frac{y^4}{2}\right)\right]$	A1	1.1b
	$=-\pi\left[\left(\frac{(0)^5}{5}+\frac{(0)^4}{2}\right)-\left(\frac{(-2)^5}{5}+\frac{(-2)^4}{2}\right)\right]$	M1	3.4
	$=1.6\pi\,\text{cm}^3$ **or** awrt 5.03 cm^3	A1	1.1b
		(4)	
			(8 marks)

Notes:

(a)

M1: Obtains x in terms of y and $\cos\theta$

M1: Obtains an equation connecting y and $\sin\theta$

M1: Uses Pythagoras to obtain an equation in x and y only

A1*: Obtains printed answer

(b)

M1: Uses the correct volume of revolution formula with the given expression

A1: Correct integration

M1: Correct use of correct limits

A1: Correct volume

Question	Scheme	Marks	AOs
8	$2+4\lambda-2(4-2\lambda)-6+\lambda=6\Rightarrow\lambda=...$	M1	1.1b
	$\lambda=2\Rightarrow$ Required point is $(2+2(4),\ 4+2(-2),\ -6+2(1))$ $(10,\ 0,\ -4)$	A1	1.1b
	$2+t-2(4-2t)-6+t=6\Rightarrow t=...$	M1	3.1a
	$t=3$ so reflection of $(2,4,-6)$ is $(2+6(1),4+6(-2),-6+6(1))$	M1	3.1a
	$(8,\ -8,\ 0)$	A1	1.1b
	$\begin{pmatrix}10\\0\\-4\end{pmatrix}-\begin{pmatrix}8\\-8\\0\end{pmatrix}=\begin{pmatrix}2\\8\\-4\end{pmatrix}$	M1	3.1a
	$\mathbf{r}=\begin{pmatrix}10\\0\\-4\end{pmatrix}+k\begin{pmatrix}1\\4\\-2\end{pmatrix}$ or equivalent e.g. $\left(\mathbf{r}-\begin{pmatrix}10\\0\\-4\end{pmatrix}\right)\times\begin{pmatrix}1\\4\\-2\end{pmatrix}=\mathbf{0}$	A1	2.5
		(7)	
			(7 marks)

Notes:

M1: Substitutes the parametric equation of the line into the equation of the plane and solves for λ

A1: Obtains the correct coordinates of the intersection of the line and the plane

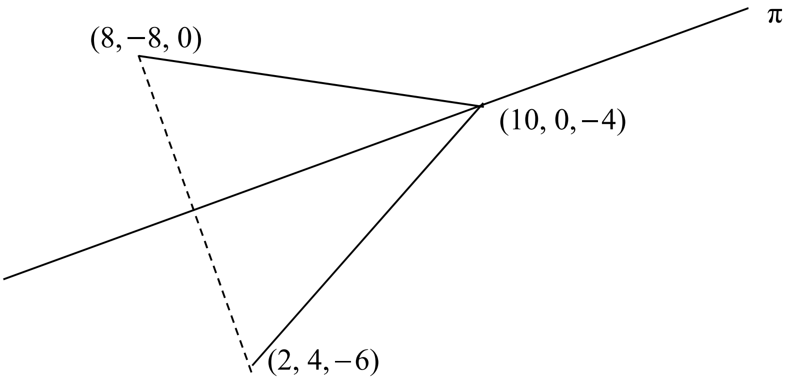
M1: Substitutes the parametric form of the line perpendicular to the plane passing through (2, 4, −6) into the equation of the plane to find t

M1: Find the reflection of (2, 4, −6) in the plane

A1: Correct coordinates

M1: Determines the direction of l by subtracting the appropriate vectors

A1: Correct vector equation using the correct notation



Question	Scheme	Marks	AOs
9(a)(i)	Weight = mass × g $\Rightarrow m = \dfrac{30\,000}{g} = 3000$ But mass is in thousands of kg, so $m = 3$	M1	3.3
(ii)	$\dfrac{\mathrm{d}x}{\mathrm{d}t} = 40\cos t + 20\sin t,\ \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -40\sin t + 20\cos t$	M1	1.1b
	$3\left(-40\sin t + 20\cos t\right) + 4\left(40\cos t + 20\sin t\right)$ $+40\sin t - 20\cos t = \ldots$	M1	1.1b
	$= 200\cos t$ so PI is $x = 40\sin t - 20\cos$	A1*	2.1
	or		
	Let $x = a\cos t + b\sin t$ $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -a\sin t + b\cos t,\ \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -a\cos t - b\sin t$	M1	1.1b
	$4b - 2a = 200,\ -2b - 4a = 0 \Rightarrow a = \ldots, b = \ldots$	M1	2.1
	$x = 40\sin t - 20\cos t$	A1*	1.1b
(iii)	$3\lambda^2 + 4\lambda + 1 = 0 \Rightarrow \lambda = -1, -\dfrac{1}{3}$	M1	1.1b
	$x = A\mathrm{e}^{-t} + B\mathrm{e}^{-\frac{1}{3}t}$	A1	1.1b
	$x = PI + CF$	M1	1.1b
	$x = A\mathrm{e}^{-t} + B\mathrm{e}^{-\frac{1}{3}t} + 40\sin t - 20\cos t$	A1	1.1b
		(8)	
(b)	$t = 0, x = 0 \Rightarrow A + B = 20$	M1	3.4
	$x = 0, \dfrac{\mathrm{d}x}{\mathrm{d}t} = -Ae^{-t} - \dfrac{1}{3}Be^{-\frac{1}{3}t} + 40\cos t + 20\sin t = 0$ $\Rightarrow A + \dfrac{1}{3}B = 40$	M1	3.4
	$x = 50\mathrm{e}^{-t} - 30\mathrm{e}^{-\frac{1}{3}t} + 40\sin t - 20\cos t$	A1	1.1b
	$t = 9 \Rightarrow x = 33\text{m}$	A1	3.4
		(4)	
		(12 marks)	

Question 9 notes:
(a)(i) **M1:** Correct explanation that in the model, $m = 3$
(ii) **M1:** Differentiates the given PI twice **M1:** Substitutes into the given differential equation **A1*:** Reaches 200cos$t$ and makes a conclusion **or** **M1:** Uses the correct form for the PI and differentiates twice **M1:** Substitutes into the given differential equation and attempts to solve **A1*:** Correct PI
(iii) **M1:** Uses the model to form and solve the auxiliary equation **A1:** Correct complementary function **M1:** Uses the correct notation for the general solution by combining PI and CF **A1:** Correct General Solution for the model
(b) **M1:** Uses the initial conditions of the model, $t = 0$ at $x = 0$, to form an equation in A and B **M1:** Uses $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ at $x = 0$ in the model to form an equation in A and B **A1:** Correct PS **A1:** Obtains 33m using the assumptions made in the model

Write your name here

Surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number ☐☐☐☐☐

Candidate Number ☐☐☐☐

Further Mathematics

Advanced
Paper 2: Core Pure Mathematics 2

Sample Assessment Material for first teaching September 2017
Time: 1 hour 30 minutes

Paper Reference
9FM0/02

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information
- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice
- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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S 5 4 4 3 9 A 0 1 1 7

Answer ALL questions. Write your answers in the spaces provided.

1. The roots of the equation

$$x^3 - 8x^2 + 28x - 32 = 0$$

are α, β and γ

Without solving the equation, find the value of

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

(ii) $(\alpha + 2)(\beta + 2)(\gamma + 2)$

(iii) $\alpha^2 + \beta^2 + \gamma^2$

(8)

Question 1 continued

(Total for Question 1 is 8 marks)

2. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(a) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1

(3)

The plane Π_2 has vector equation

$$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

where λ and μ are scalar parameters.

(b) Show that the vector $-\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is perpendicular to Π_2

(2)

(c) Show that the acute angle between Π_1 and Π_2 is 52° to the nearest degree.

(3)

Question 2 continued

(Total for Question 2 is 8 marks)

3. (i)

$$\mathbf{M} = \begin{pmatrix} 2 & a & 4 \\ 1 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

where a is a constant.

(a) For which values of a does the matrix **M** have an inverse?

(2)

Given that **M** is non-singular,

(b) find $\mathbf{M}^{-1}$ in terms of a

(4)

(ii) Prove by induction that for all positive integers n,

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$$

(6)

Question 3 continued

(Total for Question 3 is 12 marks)

4. A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad n \in \mathbb{Z}^+$$

(2)

(b) Hence, show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(5)

Question 4 continued

(Total for Question 4 is 7 marks)

5.

$$y = \sin x \sinh x$$

(a) Show that $\dfrac{d^4 y}{dx^4} = -4y$

(4)

(b) Hence find the first three non-zero terms of the Maclaurin series for y, giving each coefficient in its simplest form.

(4)

(c) Find an expression for the nth non-zero term of the Maclaurin series for y.

(2)

Question 5 continued

(Total for Question 5 is 10 marks)

6. (a) (i) Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 4 - 3\mathbf{i}| = 5$$

Taking the initial line as the positive real axis with the pole at the origin and given that $\theta \in [\alpha, \alpha + \pi]$, where $\alpha = -\arctan\left(\frac{4}{3}\right)$,

(ii) show that this locus of points can be represented by the polar curve with equation

$$r = 8\cos\theta + 6\sin\theta$$

(6)

The set of points A is defined by

$$A = \left\{z : 0 \leqslant \arg z \leqslant \frac{\pi}{3}\right\} \cap \left\{z : |z - 4 - 3\mathbf{i}| \leqslant 5\right\}$$

(b) (i) Show, by shading on your Argand diagram, the set of points A.

(ii) Find the **exact** area of the region defined by A, giving your answer in simplest form.

(7)

Question 6 continued

(Total for Question 6 is 13 marks)

7. At the start of the year 2000, a survey began of the number of foxes and rabbits on an island.

At time t years after the survey began, the number of foxes, f, and the number of rabbits, r, on the island are modelled by the differential equations

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.2f + 0.1r$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.2f + 0.4r$$

(a) Show that $\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\frac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$

(3)

(b) Find a general solution for the number of foxes on the island at time t years.

(4)

(c) Hence find a general solution for the number of rabbits on the island at time t years.

(3)

At the start of the year 2000 there were 6 foxes and 20 rabbits on the island.

(d) (i) According to this model, in which year are the rabbits predicted to die out?

(ii) According to this model, how many foxes will be on the island when the rabbits die out?

(iii) Use your answers to parts (i) and (ii) to comment on the model.

(7)

Question 7 continued

Question 7 continued

Question 7 continued

(Total for Question 7 is 17 marks)

TOTAL FOR PAPER IS 75 MARKS

Paper 2: Core Pure Mathematics 2 Mark Scheme

Question	Scheme	Marks	AOs
1(i)	$\alpha+\beta+\gamma=8,\quad \alpha\beta+\beta\gamma+\gamma\alpha=28,\quad \alpha\beta\gamma=32$	B1	3.1a
	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$	M1	1.1b
	$=\frac{7}{8}$	A1ft	1.1b
		(3)	
(ii)	$(\alpha+2)(\beta+2)(\gamma+2)=(\alpha\beta+2\alpha+2\beta+4)(\gamma+2)$	M1	1.1b
	$=\alpha\beta\gamma+2(\alpha\beta+\alpha\gamma+\beta\gamma)+4(\alpha+\beta+\gamma)+8$	A1	1.1b
	$=32+2(28)+4(8)+8=128$	A1	1.1b
		(3)	
	Alternative:		
	$(x-2)^3-8(x-2)^2+28(x-2)-32=0$	M1	1.1b
	$=\ldots-8+\ldots-32+\ldots-56-32=-128$	A1	1.1b
	$\therefore(\alpha+2)(\beta+2)(\gamma+2)=128$	A1	1.1b
		(3)	
(iii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$	M1	3.1a
	$=8^2-2(28)=8$	A1ft	1.1b
		(2)	
		(8 marks)	

Notes:

(i)

B1: Identifies the correct values for all 3 expressions (can score anywhere)

M1: Uses a correct identity

A1ft: Correct value (follow through their 8, 28 and 32)

(ii)

M1: Attempts to expand

A1: Correct expansion

A1: Correct value

Alternative:

M1: Substitutes $x-2$ for x in the given cubic

A1: Calculates the correct constant term

A1: Changes sign and so obtains the correct value

(iii)

M1: Establishes the correct identity

A1ft: Correct value (follow through their 8, 28 and 32)

Question	Scheme	Marks	AOs
2(a)	$\begin{pmatrix}3\\-4\\2\end{pmatrix}\bullet\begin{pmatrix}6\\2\\12\end{pmatrix}=18-8+24$	M1	3.1a
	$d=\dfrac{18-8+24-5}{\sqrt{3^2+4^2+2^2}}$	M1	1.1b
	$=\sqrt{29}$	A1	1.1b
		(3)	
(b)	$\begin{pmatrix}-1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\5\end{pmatrix}=\ldots$ and $\begin{pmatrix}-1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\ldots$	M1	2.1
	$\begin{pmatrix}-1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\5\end{pmatrix}=0$ and $\begin{pmatrix}-1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\-2\end{pmatrix}=0$ $\therefore\ -\mathbf{i}-3\mathbf{j}+\mathbf{k}$ is perpendicular to Π_2	A1	2.2a
		(2)	
(c)	$\begin{pmatrix}-1\\-3\\1\end{pmatrix}\bullet\begin{pmatrix}3\\-4\\2\end{pmatrix}=-3+12+2$	M1	1.1b
	$\sqrt{(-1)^2+(-3)^2+1^2}\sqrt{(3)^2+(-4)^2+2^2}\cos\theta=11$ $\Rightarrow\cos\theta=\dfrac{11}{\sqrt{(-1)^2+(-3)^2+1^2}\sqrt{(3)^2+(-4)^2+2^2}}$	M1	2.1
	So angle between planes $\theta=52^{\circ}$ *	A1*	2.4
		(3)	
		(8 marks)	

Notes:

(a)

M1: Realises the need to and so attempts the scalar product between the normal and the position vector

M1: Correct method for the perpendicular distance

A1: Correct distance

(b)

M1: Recognises the need to calculate the scalar product between the given vector and both direction vectors

A1: Obtains zero both times and makes a conclusion

(c)

M1: Calculates the scalar product between the two normal vectors

M1: Applies the scalar product formula with their 11 to find a value for cos θ

A1*: Identifies the correct angle by linking the angle between the normal and the angle between the planes

Question	Scheme	Marks	AOs
3(i)(a)	$\lvert\mathbf{M}\rvert = 2(1+2) - a(-1-1) + 4(2-1) = 0 \Rightarrow a = \ldots$	M1	2.3
	The matrix **M** has an inverse when $a \neq -5$	A1	1.1b
		(2)	
(b)	$\text{Minors}: \begin{pmatrix} 3 & -2 & 1 \\ -a-8 & 2 & a+4 \\ 4-a & -6 & -2-a \end{pmatrix}$ or $\text{Cofactors}: \begin{pmatrix} 3 & 2 & 1 \\ a+8 & 2 & -a-4 \\ 4-a & 6 & -2-a \end{pmatrix}$	B1	1.1b
	$\mathbf{M}^{-1} = \frac{1}{\lvert\mathbf{M}\rvert}\text{adj}(\mathbf{M})$	M1	1.1b
	$\mathbf{M}^{-1} = \frac{1}{2a+10}\begin{pmatrix} 3 & a+8 & 4-a \\ 2 & 2 & 6 \\ 1 & -a-4 & -2-a \end{pmatrix}$ 2 correct rows or columns. Follow through their det**M**	A1ft	1.1b
	All correct. Follow through their det**M**	A1ft	1.1b
		(4)	
(ii)	When $n = 1$, lhs $= \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$, rhs $= \begin{pmatrix} 3^1 & 0 \\ 3(3^1-1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ So the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$ so $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}$	M1	2.4
	$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k & 0 \\ 3(3^k-1) & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$	M1	2.1
	$= \begin{pmatrix} 3\times 3^k & 0 \\ 3\times 3(3^k-1)+6 & 1 \end{pmatrix}$	A1	1.1b
	$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1}-1) & 1 \end{pmatrix}$	A1	1.1b
	If the statement is true for $n = k$ then it has been shown true for $n = k + 1$ and as it is true for $n = 1$, the statement is true for all positive integers n	A1	2.4
		(6)	
			(12 marks)

Question 3 notes:
(i)(a) **M1:** Attempts determinant, equates to zero and attempts to solve for *a* in order to establish the restriction for *a* **A1:** Provides the correct condition for *a* if **M** has an inverse
(i)(b) **B1:** A correct matrix of minors or cofactors **M1:** For a complete method for the inverse **A1ft:** Two correct rows following through their determinant **A1ft:** Fully correct inverse following through their determinant
(ii) **B1:** Shows the statement is true for $n = 1$ **M1:** Assumes the statement is true for $n = k$ **M1:** Attempts to multiply the correct matrices **A1:** Correct matrix in terms of k **A1:** Correct matrix in terms of $k + 1$ **A1:** Correct complete conclusion

Question	Scheme	Marks	AOs
4(a)	$z^n + z^{-n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$	M1	2.1
	$= 2\cos n\theta$*	A1*	1.1b
		(2)	
(b)	$\left(z + z^{-1}\right)^4 = 16\cos^4\theta$	B1	2.1
	$\left(z + z^{-1}\right)^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$	M1	2.1
	$= z^4 + z^{-4} + 4\left(z^2 + z^{-2}\right) + 6$	A1	1.1b
	$= 2\cos 4\theta + 4\left(2\cos 2\theta\right) + 6$	M1	2.1
	$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$*	A1*	1.1b
		(5)	
			(7 marks)

Notes:

(a)
M1: Identifies the correct form for z^n and z^{-n} and adds to progress to the printed answer
A1*: Achieves printed answer with no errors

(b)
B1: Begins the argument by using the correct index with the result from part (a)
M1: Realises the need to find the expansion of $\left(z + z^{-1}\right)^4$
A1: Terms correctly combined
M1: Links the expansion with the result in part (a)
A1*: Achieves printed answer with no errors

Question	Scheme	Marks	AOs
5(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x \cosh x + \cos x \sinh x$	M1	1.1a
	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \cos x \cosh x + \sin x \sinh x + \cos x \cosh x - \sin x \sinh x$ $\left(= 2\cos x \cosh x\right)$	M1	1.1b
	$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 2\cos x \sinh x - 2\sin x \cosh x$	M1	1.1b
	$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -4\sinh x \sin x = -4y$ *	A1*	2.1
		(4)	
(b)	$\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_0 = 2,\ \left(\frac{\mathrm{d}^6 y}{\mathrm{d}x^6}\right)_0 = -8,\ \left(\frac{\mathrm{d}^{10} y}{\mathrm{d}x^{10}}\right)_0 = 32$	B1	3.1a
	Uses $y = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + \ldots$ with their values	M1	1.1b
	$= \frac{x^2}{2!}(2) + \frac{x^6}{6!}(-8) + \frac{x^{10}}{10!}(32)$	A1	1.1b
	$= x^2 - \frac{x^6}{90} + \frac{x^{10}}{113400}$	A1	1.1b
		(4)	
(c)	$2(-4)^{n-1}\frac{x^{4n-2}}{(4n-2)!}$	M1 A1	3.1a 2.2a
		(2)	
			(10 marks)

Notes:

(a)

M1: Realises the need to use the product rule and attempts first derivative

M1: Realises the need to use a second application of the product rule and attempts the second derivative

M1: Correct method for the third derivative

A1*: Obtains the correct 4th derivative and links this back to *y*

(b)

B1: Makes the connection with part **(a)** to establish the general pattern of derivatives and finds the correct non-zero values

M1: Correct attempt at Maclaurin series with their values

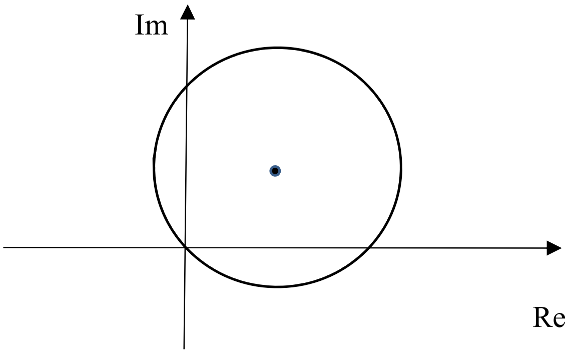
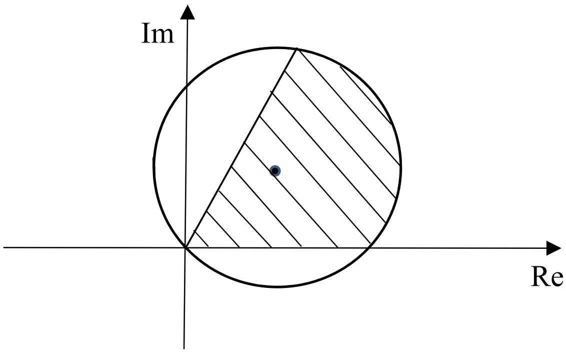
A1: Correct expression un-simplified

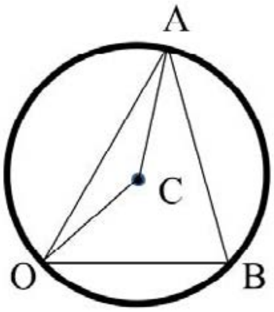
A1: Correct expression and simplified

(c)

M1: Generalising, dealing with signs, powers and factorials

A1: Correct expression

Question	Scheme	Marks	AOs
6(a)(i)		M1	1.1b
		A1	1.1b
(a)(ii)	$\lvert z-4-3\mathrm{i}\rvert=5\Rightarrow\lvert x+\mathrm{i}y-4-3\mathrm{i}\rvert=5\Rightarrow(x-4)^2+(y-3)^2=\ldots$	M1	2.1
	$(x-4)^2+(y-3)^2=25$ or any correct form	A1	1.1b
	$(r\cos\theta-4)^2+(r\sin\theta-3)^2=25$ $\Rightarrow r^2\cos^2\theta-8r\cos\theta+16+r^2\sin^2\theta-6r\sin\theta+9=25$ $\Rightarrow r^2-8r\cos\theta-6r\sin\theta=0$	M1	2.1
	$\therefore r=8\cos\theta+6\sin\theta\,*$	A1*	2.2a
		(6)	
(b)(i)		B1	1.1b
		B1ft	1.1b
(b)(ii)	$A=\frac{1}{2}\int r^2\,\mathrm{d}\theta=\frac{1}{2}\int(8\cos\theta+6\sin\theta)^2\,\mathrm{d}\theta$ $=\frac{1}{2}\int\left(64\cos^2\theta+96\sin\theta\cos\theta+36\sin^2\theta\right)\mathrm{d}\theta$	M1	3.1a
	$=\frac{1}{2}\int\left(32(\cos2\theta+1)+96\sin\theta\cos\theta+18(1-\cos2\theta)\right)\mathrm{d}\theta$	M1	1.1b
	$=\frac{1}{2}\int(14\cos2\theta+50+48\sin2\theta)\,\mathrm{d}\theta$	A1	1.1b
	$=\frac{1}{2}\left[7\sin2\theta+50\theta-24\cos2\theta\right]_0^{\frac{\pi}{3}}=\frac{1}{2}\left\{\left(\frac{7\sqrt{3}}{2}+\frac{50\pi}{3}+12\right)-(-24)\right\}$	M1	2.1
	$=\frac{7\sqrt{3}}{4}+\frac{25\pi}{3}+18$	A1	1.1b
		(7)	

Question	Scheme	Marks	AOs
	(b)(ii) Alternative:  Candidates may take a geometric approach e.g. by finding sector + 2 triangles		
	Angle $ACB = \left(\frac{2\pi}{3}\right)$ so area sector $ACB = \frac{1}{2}(5)^2\frac{2\pi}{3}$ Area of triangle $OCB = \frac{1}{2}\times 8\times 3$	M1	3.1a
	Sector area ACB + triangle area $OCB = \frac{25\pi}{3} + 12$	A1	1.1b
	Area of triangle OAC: Angle $ACO = 2\pi - \frac{2\pi}{3} - \cos^{-1}\left(\frac{5^2+5^2-8^2}{2\times5\times5}\right)$ so area $OAC = \frac{1}{2}(5)^2\sin\left(\frac{4\pi}{3} - \cos^{-1}\left(\frac{-7}{25}\right)\right)$	M1	1.1b
	$= \frac{25}{2}\left(\sin\frac{4\pi}{3}\cos\left(\cos^{-1}\left(\frac{-7}{25}\right)\right) - \cos\frac{4\pi}{3}\sin\left(\cos^{-1}\left(\frac{-7}{25}\right)\right)\right)$ $= \frac{25}{2}\left(\left(\frac{7\sqrt{3}}{50}\right) + \frac{1}{2}\sqrt{1-\left(\frac{7}{25}\right)^2}\right) = \frac{7\sqrt{3}}{4} + 6$ Total area $= \frac{25\pi}{3} + \frac{1}{2}\times 8\times 3 + 6 + \frac{7\sqrt{3}}{4}$	M1	2.1
	$= \frac{7\sqrt{3}}{4} + \frac{25\pi}{3} + 18$	A1	1.1b
			(13 marks)

Question 6 notes:

(a)(i)

M1: Draws a circle which passes through the origin

A1: Fully correct diagram

(a)(ii)

M1: Uses $z = x + \mathrm{i}y$ in the given equation and uses modulus to find equation in x and y only

A1: Correct equation in terms of x and y in any form – may be in terms of r and θ

M1: Introduces polar form, expands and uses $\cos^2\theta + \sin^2\theta = 1$ leading to a polar equation

A1*: Deduces the given equation (ignore any reference to $r = 0$ which gives a point on the curve)

(b)(i)

B1: Correct pair of rays added to their diagram

B1ft: Area between their pair of rays and inside their circle from (a) shaded, as long as there is an intersection

(b)(ii)

M1: Selects an appropriate method by linking the diagram to the polar curve in (a), evidenced by use of the polar area formula

M1: Uses double angle identities

A1: Correct integral

M1: Integrates and applies limits

A1: Correct area

(b)(ii) Alternative:

M1: Selects an appropriate method by finding angle ACB and area of sector ACB and finds area of triangle OCB to make progress towards finding the required area

A1: Correct combined area of sector ACB + triangle OCB

M1: Starts the process of finding the area of triangle OAC by calculating angle ACO and attempts area of triangle OAC

M1: Uses the addition formula to find the exact area of triangle OAC and employs a full correct method to find the area of the shaded region

A1: Correct area

Question	Scheme	Marks	AOs
7(a)	$r = 10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = 10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 2\frac{\mathrm{d}f}{\mathrm{d}t}$	M1	2.1
	$10\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 2\frac{\mathrm{d}f}{\mathrm{d}t} = -0.2f + 0.4\left(10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f\right)$	M1	2.1
	$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - 0.6\frac{\mathrm{d}f}{\mathrm{d}t} + 0.1f = 0$*	A1*	1.1b
		(3)	
(b)	$m^2 - 0.6m + 0.1 = 0 \Rightarrow m = \frac{0.6 \pm \sqrt{0.6^2 - 4 \times 0.1}}{2}$	M1	3.4
	$m = 0.3 \pm 0.1\mathrm{i}$	A1	1.1b
	$f = \mathrm{e}^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$	M1	3.4
	$f = \mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right)$	A1	1.1b
		(4)	
(c)	$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.3\mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right) + 0.1\mathrm{e}^{0.3t}\left(B\cos 0.1t - A\sin 0.1t\right)$	M1	3.4
	$r = 10\frac{\mathrm{d}f}{\mathrm{d}t} - 2f$ $= \mathrm{e}^{0.3t}\left(\left(3A + B\right)\cos 0.1t + \left(3B - A\right)\sin 0.1t\right) - 2\mathrm{e}^{0.3t}\left(A\cos 0.1t + B\sin 0.1t\right)$	M1	3.4
	$r = \mathrm{e}^{0.3t}\left(\left(A + B\right)\cos 0.1t + \left(B - A\right)\sin 0.1t\right)$	A1	1.1b
		(3)	
(d)(i)	$t = 0, f = 6 \Rightarrow A = 6$	M1	3.1b
	$t = 0, r = 20 \Rightarrow B = 14$	M1	3.3
	$r = \mathrm{e}^{0.3t}\left(20\cos 0.1t + 8\sin 0.1t\right) = 0$	M1	3.1b
	$\tan 0.1t = -2.5$	A1	1.1b
	2019	A1	3.2a
(d)(ii)	3750 foxes	B1	3.4
(d)(iii)	e.g. the model predicts a large number of foxes are on the island when the rabbits have died out and this may not be sensible	B1	3.5a
		(7)	
			(17 marks)

Question 7 notes:

(a)

M1: Attempts to differentiate the first equation with respect to *t*
M1: Proceeds to the printed answer by substituting into the second equation
A1*: Achieves the printed answer with no errors

(b)

M1: Uses the model to form and solve the auxiliary equation
A1: Correct values for *m*
M1: Uses the model to form the CF
A1: Correct CF

(c)

M1: Differentiates the expression for the number of foxes
M1: Uses this result to find an expression for the number of rabbits
A1: Correct equation

(d)(i)

M1: Realises the need to use the initial conditions in the model for the number of foxes
M1: Realises the need to use the initial conditions in the model for the number of rabbits to find both unknown constants
M1: Obtains an expression for *r* in terms of *t* and sets = 0
A1: Rearranges and obtains a correct value for tan
A1: Identifies the correct year

(d)(ii)

B1: Correct number of foxes

(d)(iii)

B1: Makes a suitable comment on the outcome of the model

Write your name here

Surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number ☐☐☐☐☐

Candidate Number ☐☐☐☐

Further Mathematics

Advanced
Further Mathematics Option 1
Paper 3: Further Pure Mathematics 1

Sample Assessment Material for first teaching September 2017

Time: 1 hour 30 minutes

Paper Reference
9FM0/3A

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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Pearson

Answer ALL questions. Write your answers in the spaces provided.

1. Use Simpson's Rule with 6 intervals to estimate

$$\int_1^4 \sqrt{1 + x^3}\, \mathrm{d}x$$

(5)

Question 1 continued

(Total for Question 1 is 5 marks)

2. Given k is a constant and that

$$y = x^3 \mathrm{e}^{kx}$$

use Leibnitz theorem to show that

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = k^{n-3}\mathrm{e}^{kx}\left(k^3x^3 + 3nk^2x^2 + 3n(n-1)kx + n(n-1)(n-2)\right)$$

(4)

Question 2 continued

(Total for Question 2 is 4 marks)

3. A vibrating spring, fixed at one end, has an external force acting on it such that the centre of the spring moves in a straight line. At time t seconds, $t \geqslant 0$, the displacement of the centre C of the spring from a fixed point O is x micrometres.

The displacement of C from O is modelled by the differential equation

$$t^2 \frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + (2 + t^2)x = t^4 \qquad \text{(I)}$$

(a) Show that the transformation $x = tv$ transforms equation (I) into the equation

$$\frac{d^2v}{dt^2} + v = t \qquad \text{(II)}$$

(5)

(b) Hence find the general equation for the displacement of C from O at time t seconds.

(7)

(c) (i) State what happens to the displacement of C from O as t becomes large.

(ii) Comment on the model with reference to this long term behaviour.

(2)

Question 3 continued

(Total for Question 3 is 14 marks)

4. $$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0 \qquad \text{(I)}$$

(a) Show that

$$\frac{d^5y}{dx^5} = ax\frac{d^4y}{dx^4} + b\frac{d^3y}{dx^3}$$

where a and b are integers to be found.

(4)

(b) Hence find a series solution, in ascending powers of x, as far as the term in x^5, of the differential equation (I) where $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$

(5)

Question 4 continued

(Total for Question 4 is 9 marks)

5. The normal to the parabola $y^2 = 4ax$ at the point $P(ap^2, 2ap)$ passes through the parabola again at the point $Q(aq^2, 2aq)$.

The line OP is perpendicular to the line OQ, where O is the origin.

Prove that $p^2 = 2$

(9)

Question 5 continued

(Total for Question 5 is 9 marks)

6. A tetrahedron has vertices $A(1, 2, 1)$, $B(0, 1, 0)$, $C(2, 1, 3)$ and $D(10, 5, 5)$.

Find

(a) a Cartesian equation of the plane ABC.

(3)

(b) the volume of the tetrahedron $ABCD$.

(3)

The plane Π has equation $2x - 3y + 3 = 0$

The point E lies on the line AC and the point F lies on the line AD.

Given that Π contains the point B, the point E and the point F,

(c) find the value of k such that $\overrightarrow{AE} = k\overrightarrow{AC}$.

(3)

Given that $\overrightarrow{AF} = \frac{1}{9}\overrightarrow{AD}$

(d) show that the volume of the tetrahedron $ABCD$ is 45 times the volume of the tetrahedron $ABEF$.

(2)

Question 6 continued

(Total for Question 6 is 11 marks)

7. P and Q are two distinct points on the ellipse described by the equation $x^2 + 4y^2 = 4$

The line l passes through the point P and the point Q.

The tangent to the ellipse at P and the tangent to the ellipse at Q intersect at the point (r, s).

Show that an equation of the line l is

$$4sy + rx = 4$$

(8)

Question 7 continued

(Total for Question 7 is 8 marks)

8.

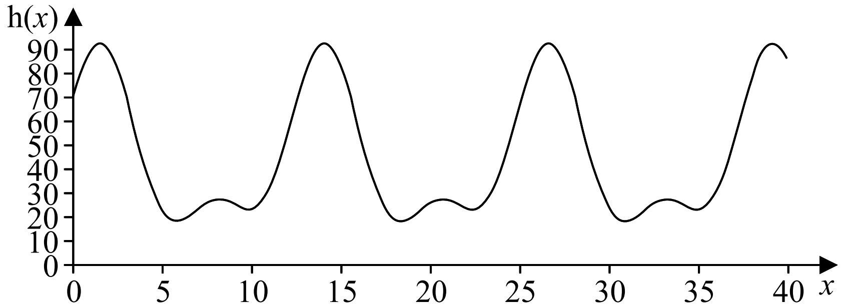


Figure 1

Figure 1 shows the graph of the function h(x) with equation

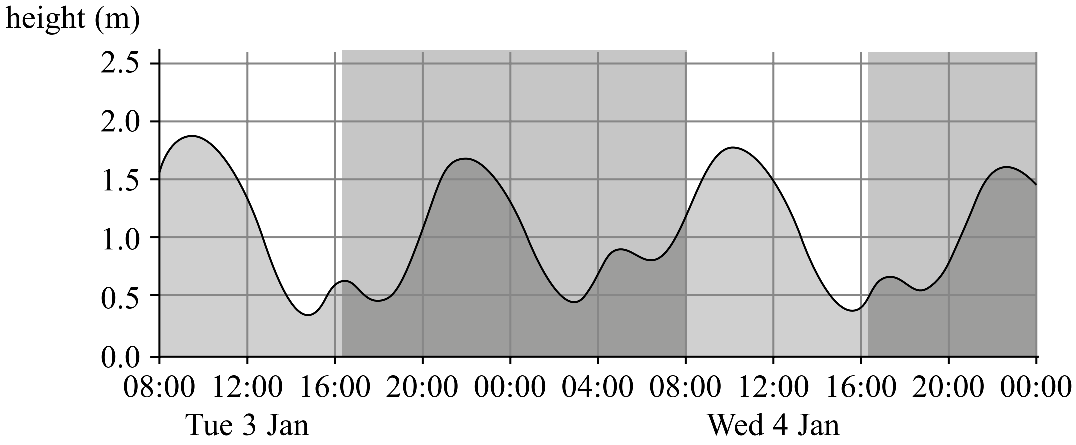
$$\mathrm{h}(x) = 45 + 15\sin x + 21\sin\left(\frac{x}{2}\right) + 25\cos\left(\frac{x}{2}\right) \qquad x \in [0, 40]$$

(a) Show that

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{(t^2 - 6t - 17)(9t^2 + 4t - 3)}{2(1 + t^2)^2}$$

where $t = \tan\left(\frac{x}{4}\right)$.

(6)



Source: [1]Data taken on 29th December 2016 from http://www.ukho.gov.uk/easytide/EasyTide

Figure 2

Figure 2 shows a graph of predicted tide heights, in metres, for Portland harbour from 08:00 on the 3rd January 2017 to the end of the 4th January 2017[1].

The graph of $k\mathrm{h}(x)$, where k is a constant and x is the number of hours after 08:00 on 3rd of January, can be used to model the predicted tide heights, in metres, for this period of time.

(b) (i) Suggest a value of k that could be used for the graph of $k\mathrm{h}(x)$ to form a suitable model.

(ii) Why may such a model be suitable to predict the times when the tide heights are at their peaks, but not to predict the heights of these peaks?

(3)

(c) Use Figure 2 and the result of part (a) to estimate, to the nearest minute, the time of the highest tide height on the 4th January 2017.

(6)

Question 8 continued

Question 8 continued

(Total for Question 8 is 15 marks)

TOTAL FOR PAPER IS 75 MARKS

Paper 3A: Further Pure Mathematics 1 Mark Scheme

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td rowspan="6">1</td><td>Step 0.5</td><td>B1</td><td>1.1b</td></tr>
<tr><td><table>
<tr><td></td><td>y_0</td><td>y_1</td><td>y_2</td><td>y_3</td><td>y_4</td><td>y_5</td><td>y_6</td></tr>
<tr><td>x</td><td>1</td><td>1.5</td><td>2</td><td>2.5</td><td>3</td><td>3.5</td><td>4</td></tr>
<tr><td>y</td><td>$\sqrt{2}$</td><td>$\sqrt{4.375}$</td><td>3</td><td>$\sqrt{16.625}$</td><td>$\sqrt{28}$</td><td>$\sqrt{43.875}$</td><td>$\sqrt{65}$</td></tr>
</table></td><td>M1</td><td>1.1b</td></tr>
<tr><td>$y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6="77.23"$</td><td>M1</td><td>1.1b</td></tr>
<tr><td>$\int_1^4 \sqrt{1+x^3}\,\mathrm{d}x \approx \frac{0.5}{3}\times"77.23"$</td><td>M1</td><td>1.1b</td></tr>
<tr><td>= 12.9</td><td>A1</td><td>1.1b</td></tr>
<tr><td></td><td>(5)</td><td></td></tr>
<tr><td colspan="4" align="right">(5 marks)</td></tr>
</table>

Notes:

B1: Use of step length 0.5

M1: Attempt to find y values with at least 2 correct

M1: Use of formula $"y_0+4y_1+2y_2+4y_3+2y_4+4y_5+y_6"$ with correct coefficients

A1: $\frac{0.5}{3}\times$ their 77.23

A1: awrt 12.9

Question	Scheme	Marks	AOs
2	$y = x^3 e^{kx}$ so $u = x^3$ and $\frac{du}{dx} = 3x^2$ and $\frac{d^2u}{dx^2} = 6x$ and $\frac{d^3u}{dx^3} = 6$ $\left(\text{and } \frac{d^4u}{dx^4} = 0\right)$	M1	1.1b
	$v = e^{kx}$ and $\frac{d^nv}{dx^n} = k^n e^{kx}$ and $\frac{d^{n-1}v}{dx^{n-1}} = k^{n-1}e^{kx}$ and $\frac{d^{n-2}v}{dx^{n-2}} = k^{n-2}e^{kx}$ (and...)	M1	2.1
	$\frac{d^ny}{dx^n} = x^3k^ne^{kx} + n3x^2k^{n-1}e^{kx} + \frac{n(n-1)}{2}6xk^{n-2}e^{kx} + \frac{n(n-1)(n-2)}{3!}6k^{n-3}e^{kx}$ and remaining terms disappear	M1	2.1
	So $\frac{d^ny}{dx^n} = k^{n-3}e^{kx}\left(k^3x^3 + 3nk^2x^2 + 3n(n-1)kx + n(n-1)(n-2)\right)$ *	A1*	1.1b
		(4)	
			(4 marks)

Notes:

M1: Differentiate $u = x^3$ three times

M1: Use $u = e^{kx}$ and establish the form of the derivatives, with at least the three shown

M1: Uses correct formula, with 2 and 3! (or 6) and with terms shown to disappear after the fourth term

A1*: Correct solution leading to the given answer stated. No errors seen

Question	Scheme	Marks	AOs
3(a)	Use of $x = tv$ to give $\frac{\mathrm{d}x}{\mathrm{d}t} = v + t\frac{\mathrm{d}v}{\mathrm{d}t}$	M1	1.1b
	Hence $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}v}{\mathrm{d}t} + t\frac{\mathrm{d}^2v}{\mathrm{d}t^2}$	M1	2.1
		A1	1.1b
	Uses $t^2\left(\text{their 2}^{\text{nd}}\text{derivative}\right) - 2t\left(\text{their 1}^{\text{st}}\text{derivative}\right) + (2+t^2)x = t^4$ and simplifies LHS	M1	2.1
	$\left(t^3\frac{\mathrm{d}^2v}{\mathrm{d}t^2} + t^3v = t^4 \text{ leading to}\right)$ $\frac{\mathrm{d}^2v}{\mathrm{d}t^2} + v = t$ *	A1*	1.1b
		(5)	
(b)	Solve $\lambda^2 + 1 = 0$ to give $\lambda^2 = -1$	M1	1.1b
	$v = A\cos t + B\sin t$	A1ft	1.1b
	Particular Integral is $v = kt + l$	B1	2.2a
	$\frac{\mathrm{d}v}{\mathrm{d}t} = k$ and $\frac{\mathrm{d}^2v}{\mathrm{d}t^2} = 0$ and solve $0 + kt + l = t$ to give $k = 1,\ l = 0$	M1	1.1b
	Solution: $v = A\cos t + B\sin t + t$	A1	1.1b
	Displacement of C from O is given by $x = tv = ...$	M1	3.4
	$x = t\left(A\cos t + B\sin t + t\right)$	A1	2.2a
		(7)	
(c)(i)	For large t, the displacement gets very large (and positive)	B1	3.2a
(ii)	Model suggests midpoint of spring moving relative to fixed point has large displacement when t is large, which is unrealistic. The spring may reach elastic limit / will break	B1	3.5a
		(2)	
			(14 marks)

Question 3 notes:	
(a)	
M1:	Uses product rule to obtain first derivative
M1:	Continues to differentiate again, with product rule and chain rule as appropriate, in order to establish the second derivative
A1:	Correct second derivative. Accept equivalent expressions
M1:	Shows clearly the substitution into the given equation in order to form the new equation and gathers like terms
A1*:	Fully correct solution leading to the given answer
(b)	Accept variations on symbols for constants throughout
M1:	Form and solve a quadratic Auxiliary Equation
A1ft:	Correct form of the Complementary Function for their solutions to the AE
B1:	Deduces the correct form for the Particular Integral (note $v = mt^2 + kt + l$ is fine)
M1:	Differentiates their Particular Integral and substitutes their derivatives into the equation to find the constants (*m* = 0 if used)
A1:	Correct general solution for equation (II)
M1:	Links the solution to equation (II) to the solution of the model equation correctly to find the displacement equation
A1:	Deduces the correct general solution for the displacement
(c)(i)	
B1:	States that for large *t* the displacement is large o.e. Accept e.g. as $t \to \infty$, $x \to \infty$
(c)(ii)	
B1:	Reflect on the context of the original problem. Accept 'model unrealistic' / 'spring will break'

Question	Scheme	Marks	AOs
4(a)	$y'' = 2xy' - y \Rightarrow y''' = 2xy'' + 2y' - y'$	M1	1.1b
		A1	1.1b
	$y''' = 2xy'' + y' \Rightarrow y'''' = 2xy''' + 2y'' + y''$	M1	2.1
	$y'''' = 2xy''' + 3y'' \;\Rightarrow\; y''''' = 2xy'''' + 5y'''$	A1	2.1
		(4)	
(b)	$x = 0, y = 0, y' = 1 \Rightarrow y''(0) = 0\pi$ from equation	B1	2.2a
	$y'''(0) = 2\times 0\times y''(0) + 1 = 1;\quad y''''(0) = 2\times 0\times 1 + 3\times 0 = 0;$ $x = 0,\ y'''(0) = 1,\ y''''(0) = 0 \Rightarrow y'''''(0) = 5$	M1	1.1b
		A1	1.1b
	$y = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 + \frac{y''''(0)}{24}x^4 + \frac{y'''''(0)}{120}x^5 + ...$	M1	2.5
	Series solution: $y = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + ...$	A1ft	1.1b
		(5)	
			(9 marks)

Notes:

(a)

M1: Attempts to differentiate equation with use of the product rule

A1: cao. Accept if terms all on one side

M1: Continues the process of differentiating to progress towards the goal. Terms may be kept on one side, but an expression in the fourth derivative should be obtained

A1: Completes the process to reach the fifth derivative and rearranges to the correct form to obtain the correct answer by correct solution only

(b)

B1: Deduces the correct value for $y''(0)$ from the information in the question

M1: Finds the values of the derivatives at the given point

A1: All correct

M1: Correct mathematical language required with given denominators. Can be in factorial form

A1ft: Correct series, must start $y = ...$ Follow through the values of their derivatives at 0

Question	Scheme	Marks	AOs
5	$y^2 = 4ax \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$	M1	2.1
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y} \Rightarrow$ Gradient of normal is $\frac{-y}{2a} = -p$	A1	1.1b
	Equation of normal is : $y - 2ap = -p(x - ap^2)$	M1	1.1b
	Normal passes through $Q(aq^2, 2aq)$ so $2aq + apq^2 = 2ap + ap^3$	M1	3.1a
	Grad OP × Grad $OQ = -1 \Rightarrow \frac{2ap}{ap^2}\frac{2aq}{aq^2} = -1$	M1	2.1
	$q = \frac{-4}{p}$	A1	1.1b
	$2a\left(\frac{-4}{p}\right) + ap\left(\frac{16}{p^2}\right) = 2ap + ap^3 \Rightarrow p^4 + 2p^2 - 8 = 0$	M1	2.1
	$(p^2 - 2)(p^2 + 4) = 0 \Rightarrow p^2 = ...$	M1	1.1b
	Hence (as $p^2 + 4 \neq 0$), $p^2 = 2$*	A1*	1.1b
		(9)	
	Alternative 1 First three marks as above and then as follows	M1	2.1
		A1	1.1b
		M1	1.1b
	Solves $y^2 = 4ax$ and their normal simultaneously to find, in terms of a and p, either $x_Q\left(= ap^2 + 4a + \frac{4a}{p^2}\right)$ or $y_Q\left(= -2ap - \frac{4a}{p}\right)$	M1	3.1a
	Finds the second coordinate of Q in terms of a and p	M1	1.1b
	Both $x_Q = ap^2 + 4a + \frac{4a}{p^2}$ and $y_Q = -2ap - \frac{4a}{p}$	A1	1.1b
	Grad OP × Grad $OQ = -1 \Rightarrow \frac{2ap}{ap^2} \times \frac{-2ap - \frac{4a}{p}}{ap^2 + 4a + \frac{4a}{p^2}} = -1$	M1	2.1
	Simplifies expression and solves: $4p^2 + 8 = p^4 + 4p^2 + 4$ $\Rightarrow p^4 - 4 = 0 \Rightarrow (p^2 - 2)(p^2 + 2) = 0 \Rightarrow p^2 = ...$	M1	2.1
	Hence (as $p^2 + 2 \neq 0$), $p^2 = 2$*	A1*	1.1b
		(9)	

Question	Scheme	Marks	AOs
5	**Alternative 2** First three marks as above and then as follows	M1 A1 M1	2.1 1.1b 1.1b
	Solves $y^2=4ax$ and their normal simultaneously to find, in terms of a and p, either $x_Q\left(=ap^2+4a+\frac{4a}{p^2}\right)$ or $y_Q\left(=-2ap-\frac{4a}{p}\right)$	M1	3.1a
	Forms a relationship between p and q from their first coordinate: **either** $y_Q=2a\left(-p-\frac{2}{p}\right)\Rightarrow q=-p-\frac{2}{p}$ **or** $x_Q=a\left(p+\frac{2}{p}\right)^2\Rightarrow q=\pm\left(p+\frac{2}{p}\right)$	M1	2.1
	$q=-p-\frac{2}{p}$ (if x coordinate used the correct root must be clearly identified before this mark is awarded)	A1	1.1b
	Grad OP × Grad $OQ=-1\Rightarrow\frac{2ap}{ap^2}\times\frac{2aq}{aq^2}=-1\ \left(\Rightarrow q=-\frac{4}{p}\right)$	M1	2.1
	Sets $q=-p-\frac{2}{p}=-\frac{4}{p}$ and solves to give $p^2=...$	M1	1.1b
	Hence $\left(\text{as } q=p+\frac{2}{p}=-\frac{4}{p}\text{ gives no solution}\right)$, $p^2=2$ (only)*	A1*	1.1b
		(9)	
		(9 marks)	

Notes:

(a)

M1: Begins proof by differentiating and using the perpendicularity condition at point P in order to find the equation of the normal

A1: Correct gradient of normal, $-p$ only

M1: Use of $y-y_1=m(x-x_1)$. Accept use of $y=mx+c$ and then substitute to find c

M1: Substitute coordinates of Q into their equation to find an equation relating p and q

M1: Use of $m_1m_2=-1$ with OP and OQ to form a second equation relating p and q

A1: $q=\frac{-4}{p}$ only

M1: Solves the simultaneous equations and cancels a from their results to obtain a quadratic equation in p^2 only

M1: Attempts to solve their quadratic in p^2. Usual rules

A1*: Correct solution leading to given answer stated. No errors seen

Question 5 notes continued:

Alternative 1:

M1A1M1: As main scheme

M1: Solves $y^2 = 4ax$ and their normal simultaneously to find one of the coordinates for Q in terms of a and p as shown

M1: Finds the second coordinate of Q in terms of a and p

A1: Both coordinates correct in terms of a and p

M1: Use of $m_1m_2 = -1$ with OP and OQ. i.e. $\frac{2ap}{ap^2} \times \frac{\text{their } y_Q}{\text{their } x_Q} = -1$ with coordinates of P and their expressions for x_Q and y_Q

M1: Cancels the a's, simplifies to a quadratic in p^2 and solves the quadratic. Usual rules

A1*: Correct solution leading to the given answer stated. No errors seen

Alternative 2:

M1A1M1: As main scheme

M1: Solves $y^2 = 4ax$ and their normal simultaneously to find one of the coordinates for Q in terms of a and p as shown

M1: Uses their coordinate to form a relationship between p and q. Allow $q = \left(p + \frac{2}{p}\right)$ for this mark

A1: For $q = -p - \frac{2}{p}$. If the x coordinate was used to find q then consideration of the negative root is needed for this mark. Allow for $q = \pm\left(p + \frac{2}{p}\right)$

M1: Use of $m_1m_2 = -1$ with OP and OQ to form a second equation relating p and q only

M1: Equates expressions for q and attempts to solve to give $p^2 = ...$

A1*: Correct solution leading to the given answer stated. No errors seen. If x coordinate used, invalid solution must be rejected

Question	Scheme	Marks	AOs
6(a)	$\mathbf{AB}\times\mathbf{AC}=\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\times\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}-2-1\\-1+2\\1+1\end{pmatrix}=\begin{pmatrix}-3\\1\\2\end{pmatrix}$	M1	1.1b
	$\mathbf{r}.\begin{pmatrix}-3\\1\\2\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}.\begin{pmatrix}-3\\1\\2\end{pmatrix}=1$	M1	1.1b
	Hence $-3x+y+2z=1$	A1	1.1b
		(3)	
(b)	Volume of Tetrahedron = $\frac{1}{6}\left\|\mathbf{n}.(\mathbf{AD})\right\|$	M1	3.1a
	$=\frac{1}{6}\left\|\begin{pmatrix}-3\\1\\2\end{pmatrix}.\left(\begin{pmatrix}10\\5\\5\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}\right)\right\|$	M1	1.1b
	$=\frac{1}{6}\left\|(-27+3+8)\right\|=\frac{8}{3}$	A1	1.1b
		(3)	
(c)	$\mathbf{AE}=k\mathbf{AC}$ so E is $(1+k,\ 2-k,\ 1+2k)$	M1	3.1a
	E lies on plane so $2(1+k)-3(2-k)+3=0$, leading to $k=\ldots$	M1	3.1a
	Hence $k=\frac{1}{5}$	A1	1.1b
		(3)	
(d)	Volume $ABEF$ = $\frac{1}{6}(\mathbf{AB}\times\mathbf{AE}).\mathbf{AF}=\frac{1}{6}\left(\mathbf{AB}\times\frac{1}{5}\mathbf{AC}\right).\frac{1}{9}\mathbf{AD}$	M1	3.1a
	$=\frac{1}{45}\left(\frac{1}{6}(\mathbf{AB}\times\mathbf{AC}).\mathbf{AD}\right)$ and hence result *	A1*	2.2a
		(2)	
		(11 marks)	

Question 6 notes:
(a) **M1:** Attempting a suitable cross product. Accept use of unit vectors **M1:** Complete method that would lead to finding the Cartesian equation of plane **A1:** Accept any equivalent form
(b) **M1:** Identifies suitable vectors and attempts to substitute into a correct formula. Accept use of unit vectors **M1:** Correct form of scalar triple product using their **n** from part (a) **A1:** $\frac{8}{3}$ or exact equivalent form
(c) **M1:** Uses that *E* is on *AC* in order to find an expression for *E* **M1:** Uses that *E* is in the plane Π to form and solve an expression in *k* **A1:** $\frac{1}{5}$ o.e. only
(d) **M1:** Uses formula for volume of tetrahedron and substitutes for $\mathbf{AE}$ and $\mathbf{AF}$ **A1*:** Deduces result: Use of $\frac{1}{6}\left(\mathbf{AB} \times \mathbf{AC}\right).\mathbf{AD}$ is required and no errors seen in solution

Question	Scheme	Marks	AOs
7	$x^2+4y^2=4 \Rightarrow 2x+8y\frac{\mathrm{d}y}{\mathrm{d}x}=0 \quad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=...$	M1	3.1a
	Equation of tangent at $P(x_1, y_1)$ is $(y-y_1) = -\frac{x_1}{4y_1}(x-x_1)$	M1	3.1a
	$xx_1+4yy_1 = x_1^2+4y_1^2 = 4$ and at $Q(x_2, y_2)$: $xx_2+4yy_2=4$	A1	2.2a
	Intersect at (r, s) gives $rx_1+4sy_1 = 4$ and $rx_2+4sy_2 = 4$	B1	2.1
	Uses their previous results to find the gradient of the line *l*	M1	3.1a
	$\frac{y_2-y_1}{x_2-x_1} = \frac{-r}{4s}$	A1	1.1b
	Equation of *l* is $y-y_1=\frac{-r}{4s}(x-x_1)$	M1	2.1
	$4sy+rx = 4sy_1+rx_1 = 4$ *	A1*	2.2a
		(8)	
		(8 marks)	

Notes:

M1: Attempts to solve the problem by using differentiation to obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$

M1: Realise the need to form a general equation of the tangent at (x_1, y_1). May use alternative variables

A1: Deduces $x_1^2+4y_1^2=4$ to obtain a correct equation and deduces a correct second equation

B1: Uses (r, s) in both equations to form the two given equations or exact equivalents

M1: Uses their previous results to find the gradient of the line *l*

A1: $\frac{-r}{4s}$

M1: Formulates the line *l* with their $\frac{-r}{4s}$. Use of $y-y_1=m(x-x_1)$ or $y=mx+c$ with their gradient and an attempt to find c

A1*: Correct solution leading to $4sy+rx=4sy_1+rx_1$ with deduction that this equals 4 as (x_1, y_1) is on the ellipse. No errors seen

Question	Scheme	Marks	AOs
8(a)	$\mathrm{h}(x) = 45 + 15\sin x + 21\sin\left(\frac{x}{2}\right) + 25\cos\left(\frac{x}{2}\right)$		
	$\frac{\mathrm{dh}}{\mathrm{d}x} = 15\cos x + \frac{21}{2}\cos\left(\frac{x}{2}\right) - \frac{25}{2}\sin\left(\frac{x}{2}\right)$	M1	1.1b
	$\frac{\mathrm{dh}}{\mathrm{d}x} = ... + .. \frac{1-t^2}{1+t^2} - .. \frac{2t}{1+t^2}$	M1	1.1a
	e.g. $\frac{\mathrm{d}h}{\mathrm{d}x} = \; .. \left(2\left(\frac{1-t^2}{1+t^2}\right)^2 - 1\right) + ...$ **or** $\frac{\mathrm{d}h}{\mathrm{d}x} = \; .. \frac{1-\left(\frac{2t}{1-t^2}\right)^2}{1+\left(\frac{2t}{1-t^2}\right)^2} + ...$	M1	3.1a
	e.g. $\frac{\mathrm{d}h}{\mathrm{d}x} = 15\left(2\left(\frac{1-t^2}{1+t^2}\right)^2 - 1\right) + \frac{21}{2}\left(\frac{1-t^2}{1+t^2}\right) - \frac{25}{2}\left(\frac{2t}{1+t^2}\right)$	A1	1.1b
	$\ldots = \frac{15[4(1-t^2)^2 - 2(1+t^2)^2] + 21(1-t^2)(1+t^2) - 50t(1+t^2)}{2(1+t^2)^2}$ x	M1	2.1
	$\ldots = \frac{9t^4 - 50t^3 - 180t^2 - 50t + 51}{2(1+t^2)^2} = \frac{(t^2-6t-17)(9t^2+4t-3)}{2(1+t^2)^2}$ *	A1*	2.1
		(6)	
	8(a) Alternative $\mathrm{h}(x) = ... + 21\left(\frac{2t}{1+t^2}\right) + 25\left(\frac{1-t^2}{1+t^2}\right)$	M1	1.1a
	$= ... + 15\left[2\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)\right] + ...$ **or** $= ... + 15\left(\frac{2\left(\frac{2t}{1-t^2}\right)}{1+\left(\frac{2t}{1-t^2}\right)^2}\right) + ...$	M1	2.1
	$\mathrm{h}(x) = 45 + \frac{15(4t(1-t^2)) + 42t(1+t^2) + 25(1-t^4)}{(1+t^2)^2}$	M1	1.1b
	$\mathrm{h}(x) = 45 - \frac{25t^4 + 18t^3 - 102t - 25}{(1+t^2)^2}$ or $\frac{20t^4 - 18t^3 + 90t^2 + 102t + 70}{(1+t^2)^2}$	A1	1.1b
	$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}h}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{('u')(1+t^2)^2 - ('u')(4t(1+t^2))}{(1+t^2)^4} \times \frac{1}{4}(1+t^2)$	M1	3.1a
	$... = \frac{9t^4 - 50t^3 - 180t^2 - 50t + 51}{2(1+t^2)^2} = \frac{(t^2-6t-17)(9t^2+4t-3)}{2(1+t^2)^2}$ *	A1*	2.1
		(6)	

Question	Scheme	Marks	AOs
8(b)(i)	Accept any value between $\frac{1}{40} = 0.025$ and $\frac{1}{60} \approx 0.167$ inclusive	B1	3.3
(ii)	Suitable for times since the graphs both oscillate bi-modally with about the same periodicity	B1	3.4
	Not suitable for predicting heights since the heights of the peaks vary over time, but the graph of h(x) has fixed peak height	B1	3.5b
		(3)	
8(c)	Solves at least one of the quadratics $t = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 17}}{2} = 3 \pm \sqrt{26}$ or $t = \frac{-4 \pm \sqrt{16 - 4 \times 9 \times (-3)}}{18} = \frac{-2 \pm \sqrt{31}}{9}$	**M1**	1.1b
	Finds corresponding x values, $x = 4\tan^{-1}(t)$ for at least one value of t from the $9t^2 + 4t - 3$ factor	**M1**	1.1b
	One correct value for these x e.g. $x =$ awrt -2.797 or 9.770,1.510	**A1**	1.1b
	Maximum peak height occurs at smallest positive value of x, from first graph, but the third of these peaks needed, So $t = 1.509\ldots + 8\pi = 26.642$ is the is the required time	**M1**	3.4
	$x = 26.642$ corresponds to 26 hours and 39 minutes (nearest minute) after 08:00 on 3rd January (Allow if a different greatest peak height used)	**M1**	3.4
	Time of greatest tide height is approximately 10:39 (am) (also allow 10:38 or 10:40)	**A1**	3.2a
		(6)	
		(15 marks)	

Notes:

(a)

M1: Differentiates h(x)

M1: Applies t-substitution to both $\left(\frac{x}{2}\right)$ terms with their coefficients

M1: Forms a correct expression in t for the cos x term, using double angle formula and t-substitution, or double 't'-substitution

A1: Fully correct expression in t for $\frac{\mathrm{d}h}{\mathrm{d}x}$

M1: Gets all terms over the correct common factor. Numerators must be appropriate for their terms

A1*: Achieves the correct answer via expression with correct quartic numerator before factorisation

Question 8 notes continued:

Alternative:

(a)

M1: Applies t-substitution to both $\left(\frac{x}{2}\right)$ terms

M1: Forms a correct expression in t for the $\sin x$ term, using double angle formula and t-substitution, or double 't'-substitution

M1: Gets all terms in t over the correct common factor. Numerators must be appropriate for their terms. May include the constant term too

A1: Fully correct expression in t for $\mathrm{h}(x)$

M1: Differentiates, using both chain rule and quotient rule with their 'u'

A1*: Achieves the correct answer via expression with correct quartic numerator before factorisation

Note: The individual terms may be differentiated before putting over a common denominator. In this case score the third M for differentiating with chain rule and quotient rule, then r return to the original scheme

(b)(i)

B1: Any value between $\frac{1}{40}$ (e.g. taking $\mathrm{h}(0)$ as reference point) **or** $\frac{1}{60}$ (taking lower peaks as reference)

NB: **Taking high peak as reference gives $\frac{1}{50}$**

(b)(ii)

B1: Should mention both the bimodal nature and periodicity for the actual data match the graph of h

B1: Mentions that the heights of peaks vary in each oscillation

(c)

M1: Solves (at least) one of the quadratic equations in the numerator

M1: Must be attempting to solve the quadratic factor from which the solution comes $9t^2 + 4t - 3$ and using $t = \tan\left(\frac{x}{4}\right)$ to find a corresponding value for x

A1: At least one correct x value from solving the requisite quadratic: awrt any of – 2.797, 1.510, 9.770, 14.076, 22.336, 26.642, 34.902 or 39.208

M1: Uses graph of h to pick out their $x = 26.642$ as the time corresponding to the third of the higher peaks, which is the highest of the peaks on 4th January on the tide height graph. As per scheme or allow if all times listed and correct one picked

M1: Finds the time for one of the values of t corresponding to the highest peaks. E.g. 1.5096...~ 09:31 (3rd January) or 14.076... ~ 22:05 (3rd January) or 26.642…~ 10:39 (4th January) or 39.208…~ 23:13 (4th January). (Only follow through on use of the smallest positive t solution $+\ 4k\pi$)

A1: Time of greatest tide height on 4th January is approximately 10:39. Also allow 10:38 or 10:40

Write your name here

Surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number ☐☐☐☐☐

Candidate Number ☐☐☐☐

Further Mathematics

Advanced
Further Mathematics Option 2
Paper 4: Further Pure Mathematics 2

Sample Assessment Material for first teaching September 2017

Time: 1 hour 30 minutes

Paper Reference
9FM0/4A

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B)
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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1/1/1/1/1/

S 5 4 4 4 4 A 0 1 2 9

Answer ALL questions. Write your answers in the spaces provided.

1. (i) Use the Euclidean algorithm to find the highest common factor of 602 and 161.

Show each step of the algorithm.

(3)

(ii) The digits which can be used in a security code are the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Originally the code used consisted of two distinct odd digits, followed by three distinct even digits.

To enable more codes to be generated, a new system is devised. This uses two distinct even digits, followed by any three other distinct digits. No digits are repeated.

Find the increase in the number of possible codes which results from using the new system.

(4)

Question 1 continued

(Total for Question 1 is 7 marks)

2. A transformation from the z-plane to the w-plane is given by

$$w = z^2$$

(a) Show that the line with equation $\mathrm{Im}(z) = 1$ in the z-plane is mapped to a parabola in the w-plane, giving an equation for this parabola.

(4)

(b) Sketch the parabola on an Argand diagram.

(2)

Question 2 continued

(Total for Question 2 is 6 marks)

3. The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

(a) Show that 4 is an eigenvalue of **M**, and find the other two eigenvalues.

(4)

(b) For each of the eigenvalues find a corresponding eigenvector.

(4)

(c) Find a matrix **P** such that $\mathbf{P}^{-1}\mathbf{MP}$ is a diagonal matrix.

(2)

Question 3 continued

Question 3 continued

Question 3 continued

(Total for Question 3 is 10 marks)

4. (i) A group G contains distinct elements a, b and e where e is the identity element and the group operation is multiplication.

Given $a^2b = ba$, prove $ab \neq ba$

(4)

(ii) The set $H = \{1, 2, 4, 7, 8, 11, 13, 14\}$ forms a group under the operation of multiplication modulo 15

(a) Find the order of each element of H.

(3)

(b) Find three subgroups of H each of order 4, and describe each of these subgroups.

(4)

The elements of another group J are the matrices $\begin{pmatrix} \cos\left(\frac{k\pi}{4}\right) & \sin\left(\frac{k\pi}{4}\right) \\ -\sin\left(\frac{k\pi}{4}\right) & \cos\left(\frac{k\pi}{4}\right) \end{pmatrix}$

where $k =1, 2, 3, 4, 5, 6, 7, 8$ and the group operation is matrix multiplication.

(c) Determine whether H and J are isomorphic, giving a reason for your answer.

(2)

Question 4 continued

Question 4 continued

Question 4 continued

(Total for Question 4 is 13 marks)

5.

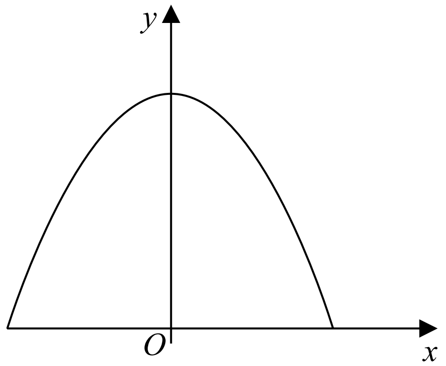


Figure 1

An engineering student makes a miniature arch as part of the design for a piece of coursework.

The cross-section of this arch is modelled by the curve with equation

$$y = A - \frac{1}{2}\cosh 2x, \qquad -\ln a \leqslant x \leqslant \ln a$$

where $a > 1$ and A is a positive constant. The curve begins and ends on the x-axis, as shown in Figure 1.

(a) Show that the length of this curve is $k\left(a^2 - \frac{1}{a^2}\right)$, stating the value of the constant k.

(5)

The length of the curved cross-section of the miniature arch is required to be 2 m long.

(b) Find the height of the arch, according to this model, giving your answer to 2 significant figures.

(4)

(c) Find also the width of the base of the arch giving your answer to 2 significant figures.

(1)

(d) Give the equation of another curve that could be used as a suitable model for the cross-section of an arch, with approximately the same height and width as you found using the first model.
(You do not need to consider the arc length of your curve)

(2)

Question 5 continued

Question 5 continued

Question 5 continued

(Total for Question 5 is 12 marks)

6. A curve has equation

$$|z + 6| = 2\,|z - 6| \qquad z \in \mathbb{C}$$

(a) Show that the curve is a circle with equation $x^2 + y^2 - 20x + 36 = 0$

(2)

(b) Sketch the curve on an Argand diagram.

(2)

The line l has equation $az^* + a^*z = 0$, where $a \in \mathbb{C}$ and $z \in \mathbb{C}$

Given that the line l is a tangent to the curve and that $\arg a = \theta$

(c) find the possible values of $\tan\theta$

(5)

Question 6 continued

Question 6 continued

Question 6 continued

(Total for Question 6 is 9 marks)

7.

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \qquad n \geqslant 0$$

(a) Prove that, for $n \geqslant 2$,

$$nI_n = (n-1)I_{n-2}$$

(4)

(b)

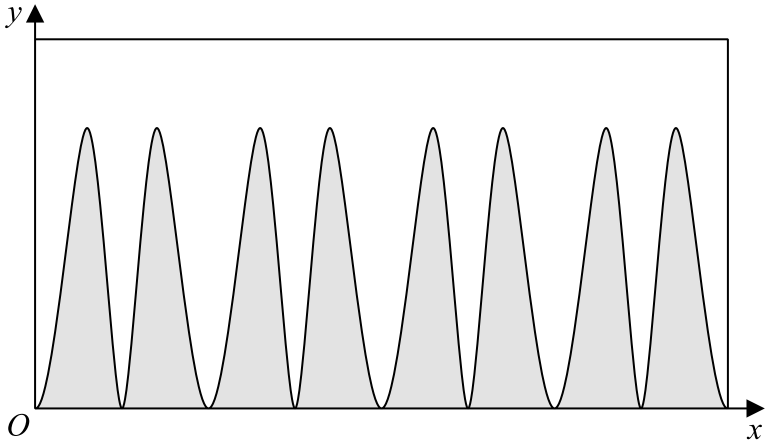


Figure 2

A designer is asked to produce a poster to completely cover the curved surface area of a solid cylinder which has diameter 1 m and height 0.7 m.

He uses a large sheet of paper with height 0.7 m and width of π m.

Figure 2 shows the first stage of the design, where the poster is divided into two sections by a curve.

The curve is given by the equation

$$y = \sin^2(4x) - \sin^{10}(4x)$$

relative to axes taken along the bottom and left hand edge of the paper.

The region of the poster below the curve is shaded and the region above the curve remains unshaded, as shown in Figure 2.

Find the exact area of the poster which is shaded.

(5)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 7 continued

Question 7 continued

Question 7 continued

(Total for Question 7 is 9 marks)

8. A staircase has n steps. A tourist moves from the bottom (step zero) to the top (step n). At each move up the staircase she can go up either one step or two steps, and her overall climb up the staircase is a combination of such moves.

If u_n is the number of ways that the tourist can climb up a staircase with n steps,

(a) explain why u_n satisfies the recurrence relation

$$u_n = u_{n-1} + u_{n-2}, \text{ with } u_1 = 1 \text{ and } u_2 = 2$$

(3)

(b) Find the number of ways in which she can climb up a staircase when there are eight steps.

(1)

A staircase at a certain tourist attraction has 400 steps.

(c) Show that the number of ways in which she could climb up to the top of this staircase is given by

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{401} - \left(\frac{1-\sqrt{5}}{2}\right)^{401}\right]$$

(5)

Question 8 continued

Question 8 continued

Question 8 continued

(Total for Question 8 is 9 marks)

TOTAL FOR PAPER IS 75 MARKS

Paper 4A: Further Pure Mathematics 2 Mark Scheme

Question	Scheme	Marks	AOs
1(i)	$602 = 3\times161+119$	M1	1.1b
	$161 = 119 + 42,\ 119 = 2\times42+35$	M1	1.1b
	$42 = 35 + 7,\ 35 = 5\times7,\ \text{hcf} = 7$	A1	1.1b
		(3)	
(ii)	Number of codes under old system = $5\times4\times4\times3\times2\ (= 480)$	B1	3.1b
	Number of codes under new system = $4\times3\times7\times6\times5\ (= 2520)$	B1	3.1b
	Subtracts first answer from second	M1	1.1b
	Increase in number of codes is 2040	A1	1.1b
		(4)	
			(7 marks)

Notes:

(i)

M1: Attempts Euclid's algorithm – (there may be an arithmetic slip finding 119)

M1: Uses Euclid's algorithm a further two times with 161 and "their 119" and then with "their 119" and "their 42"

A1: This should be accurate with all the steps shown

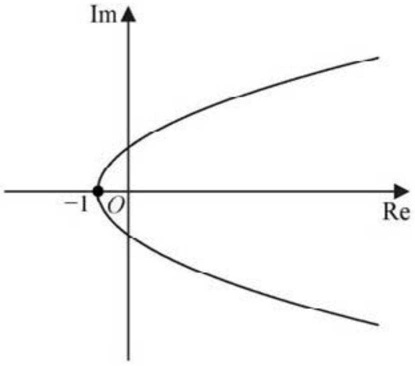
(ii)

B1: Correctly interprets the problem and uses the five odd digits and four even digits to form a correct product

B1: Interprets the new situation using the four even digits, then the seven digits which have not been used, to form a correct product

M1: Subtracts one answer from the other

A1: Correct answer

Question	Scheme	Marks	AOs
2(a)	Let $z = x + \mathrm{i}$	M1	2.1
	$w=(x+\mathrm{i})^2=(x^2-1)+2x\mathrm{i}$	A1	1.1b
	Let $w = u + \mathrm{i}v$, then $u=(x^2-1)$ and $v=2x$	M1	2.1
	$\Rightarrow v^2=4(u+1)$, which therefore represents a parabola	A1ft	2.2a
		(4)	
(b)	 M1: Sketches a parabola with symmetry about the real axis A1: Accurate sketch	M1	1.1b
		A1	1.1b
		(2)	
		(6 marks)	

Notes:

(a)
M1: Translates the information that Im(z) = 1 into a cartesian form; e.g. $z = x + \mathrm{i}$
A1: Obtains a correct expression for w
M1: Separates the real and imaginary parts and equates to u and v respectively
A1ft: Obtains a quadratic equation and states that their quadratic equation represents a parabola

(b)
M1: Sketches a parabola with symmetry about the real axis
A1: Accurate sketch

Question	Scheme		Marks	AOs
3(a)	Finds the characteristic equation $(2-\lambda)^2(4-\lambda)-(4-\lambda)=0$		M1	2.1
	So $(4-\lambda)(\lambda^2-4\lambda+3)=0$ so $\lambda=4$ *		A1*	2.2a
	Solves quadratic equation to give		M1	1.1b
	$\lambda=1$ and $\lambda=3$		A1	1.1b
			(4)	
(b)	Uses a correct method to find an eigenvector		M1	1.1b
	Obtains a vector parallel to one of $\begin{pmatrix}0\\0\\1\end{pmatrix}$ or $\begin{pmatrix}1\\1\\1\end{pmatrix}$ or $\begin{pmatrix}3\\-3\\1\end{pmatrix}$		A1	1.1b
	Obtains two correct vectors		A1	1.1b
	Obtains all three correct vectors		A1	1.1b
			(4)	
(c)	Uses their three vectors to form a matrix		M1	1.2
	$\begin{pmatrix}0&1&3\\0&1&-3\\1&1&1\end{pmatrix}$	**or** other correct answer with columns in a different order	A1	1.1b
			(2)	
				(10 marks)

Notes:

(a)
M1: Attempts to find the characteristic equation (there may be one slip)
A1*: Deduces that $\lambda=4$ is a solution by the method shown or by checking that $\lambda=4$ satisfies the characteristic equation
M1: Solves their quadratic equation
A1: Obtains the two correct answers as shown above

(b)
M1: Uses a correct method to find an eigenvector
A1: Obtains one correct vector (may be a multiple of the given vectors)
A1: Obtains two correct vectors (may be multiples of the given vectors)
A1: Obtains all three correct vectors (may be multiples of the given vectors)

(c)
M1: Forms a matrix with their vectors as columns

A1: $\begin{pmatrix}0&1&3\\0&1&-3\\1&1&1\end{pmatrix}$ **or** $\begin{pmatrix}1&0&3\\1&0&-3\\1&1&1\end{pmatrix}$ **or** $\begin{pmatrix}3&1&0\\-3&1&0\\1&1&1\end{pmatrix}$ **or** other correct alternative

Question	Scheme	Marks	AOs
4(i)	If we assume $ab = ba$; as $a^2b = ba$ then $ab = a^2b$	M1	2.1
	So $a^{-1}abb^{-1} = a^{-1}a^2bb^{-1}$	M1	2.1
	So $e=a$	A1	2.2a
	But this is a contradiction, as the elements e and a are distinct so $ab \neq ba$	A1	2.4
		(4)	
(ii)(a)	2 has order 4 and 4 has order 2	M1	1.1b
	7, 8 and 13 have order 4	A1	1.1b
	11 and 14 have order 2 and 1 has order 1	A1	1.1b
		(3)	
(ii)(b)	Finds the subgroup {1, 2, 4, 8} or the subgroup {1, 7, 4, 13}	M1	1.1b
	Finds both and refers to them as cyclic groups, or gives generator 2 and generator 7	A1	2.4
	Finds {1, 4, 11, 14}	B1	2.2a
	States each element has order 2 or refers to it as Klein Group	B1	2.5
		(4)	
(ii)(c)	*J* has an element of order 8, (*H* does not) or *J* is a cyclic group (*H* is not) or other valid reason	M1	2.4
	They are not isomorphic	A1	2.2a
		(2)	
			(13 marks)

Question 4 notes:
(i) **M1:** Proof begins with assumption that $ab = ba$ and deduces that this implies $ab = a^2b$ **M1:** A correct proof with working shown follows, and may be done in two stages **A1:** Concludes that assumption implies that $e=a$ **A1:** Explains clearly that this is a contradiction, as the elements *e* and *a* are distinct so $ab \neq ba$
(ii)(a) **M1:** Obtains two correct orders (usually the two in the scheme) **A1:** Finds another three correctly **A1:** Finds the final three so that all eight are correct
(ii)(b) **M1:** Finds one of the cyclic subgroups **A1:** Finds both subgroups and explains that they are cyclic groups, or gives generators 2 and 7 **B1:** Finds the non cyclic group **B1:** Uses correct terms that each element has order 2 or refers to it as Klein Group
(ii)(c) **M1:** Clearly explains how *J* differs from *H* **A1:** Correct deduction

Question	Scheme	Marks	AOs
5(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sinh 2x$	B1	2.1
	So $S = \int \sqrt{1+\sinh^2 2x}\,\mathrm{d}x$	M1	2.1
	$\therefore s = \int \cosh 2x\mathrm{d}x$	A1	1.1b
	$= \left[\frac{1}{2}\sinh 2x\right]_{-\ln a}^{\ln a}$ or $\left[\sinh 2x\right]_{0}^{\ln a}$	M1	2.1
	$= \sinh 2\ln a = \frac{1}{2}[\mathrm{e}^{2\ln a} - \mathrm{e}^{-2\ln a}] = \frac{1}{2}\left(a^2 - \frac{1}{a^2}\right)$ (so $k = ½$)	A1	1.1b
		(5)	
(b)	$\frac{1}{2}\left(a^2 - \frac{1}{a^2}\right) = 2$ so $a^4 - 4a^2 - 1 = 0$	M1	1.1b
	$a^2 = 2+\sqrt{5}$ (and $a = 2.06$ (approx.))	M1	1.1b
	When $x = \ln a$, $y = 0$ so $A = \frac{1}{2}\cosh(2\ln a)$	M1	3.4
	Height = $A - 0.5$ = awrt 0.62m	A1	1.1b
		(4)	
(c)	The width of the base = 2ln*a* = 1.4m	B1	3.4
		(1)	
(d)	A parabola of the form $y = 0.62 - 1.19\,x^2$, or other symmetric curve with its equation e.g. 0.62cos(2.2*x*)	M1A1	3.3 3.3
		(2)	
		(12 marks)	

Notes:

(a)

B1: Starts explanation by finding the correct derivative

M1: Uses their derivative in the formula for arc length

A1: Uses suitable identity to simplify the integrand and to obtain the expression in scheme

M1: Integrates and uses appropriate limits to find the required arc length

A1: Uses the definition of sinh to complete the proof and identifies the value for *k*

(b)

M1: Uses the formula obtained from the model and the length of the arch to create a quartic equation

M1: Continues to use this model to obtain a quadratic and to obtain values for *a*

M1: Attempts to find a value for *A* in order to find *h*

A1: Finds a value for the height correct to 2sf (or accept exact answer)

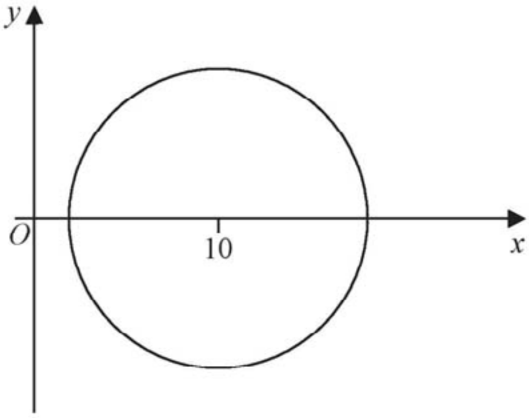
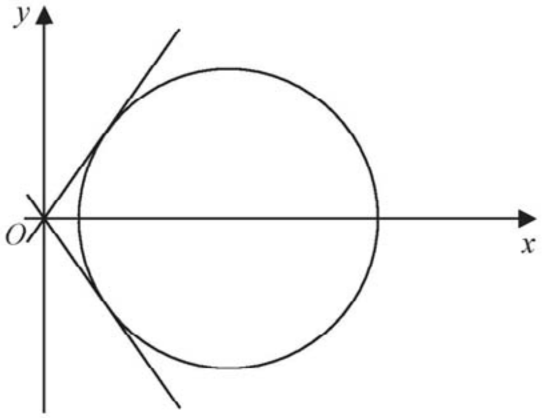
(c)

B1: Finds width to 2 sf i.e. 1.4m

(d)

M1: Chooses or describes an even function with maximum point on the *y* axis

A1: Gives suitable equation passing through (0, 0.62) and (0.7, 0) and (- 0.7, 0)

Question	Scheme	Marks	AOs
6(a)	$(x+6)^2 + y^2 = 4[(x-6)^{+}y^2]$	M1	2.1
	$x^2+y^2-20x+36=0$ which is the equation of a circle*	A1*	2.2a
		(2)	
(b)		M1	1.1b
		A1	1.1b
		(2)	
(c)	Let $a = c + \mathrm{i}d$ and $a^* = c - \mathrm{i}d$ then $(c + \mathrm{i}d)(x - \mathrm{i}y) + (c - \mathrm{i}d)(x + \mathrm{i}y) = 0$	M1	3.1a
	So $y = -\frac{c}{d}x$	A1	1.1b
	 The gradients of the tangents (from geometry) are $\pm\frac{4}{3}$	B1	3.1a
	So $-\frac{c}{d} = \pm\frac{4}{3}$ and $\frac{d}{c} = \mp\frac{3}{4}$	M1	3.1a
	So $\tan\theta = \pm\frac{3}{4}$	A1	1.1b
		(5)	

Question 6 notes:

(a)

M1: Obtains an equation in terms of x and y using the given information

A1*: Expands and simplifies the algebra, collecting terms and obtains a circle equation correctly, deducing that this is a circle

(b)

M1: Draws a circle with centre at (10, 0)

A1: (Radius is 8) so circle does not cross the y axis

(c)

M1: Attempts to convert line equation into a cartesian form

A1: Obtains a simplified line equation

B1: Uses geometry to deduce the gradients of the tangents

M1: Understands the connection between arg a and the gradient of the tangents and uses this connection

A1: Correct answers

Question	Scheme	Marks	AOs
7(a)	$I_n = \int_0^{\frac{\pi}{2}} \sin x \sin^{n-1} x \,\mathrm{d}x$	M1	2.1
	$= \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} - (-)\int_0^{\frac{\pi}{2}} \cos^2 x(n-1)\sin^{n-2} x\,\mathrm{d}x$	A1	1.1b
	Obtains $= 0-(-)\int_0^{\frac{\pi}{2}} (1-\sin^2 x)(n-1)\sin^{n-2} x\,\mathrm{d}x$	M1	1.1b
	So $I_n = (n-1)I_{n-2} - (n-1)I_n$ and hence $nI_n = (n-1)I_{n-2}$ *	A1*	2.1
		(4)	
(b)	uses $I_n = \frac{(n-1)}{n} I_{n-2}$ to give $I_{10} = \frac{9}{10} I_8$ or $I_2 = \frac{1}{2} I_0$	M1	3.1b
	So $I_{10} = \frac{9\times 7\times 5\times 3\times 1}{10\times 8\times 6\times 4\times 2} I_0$ x	M1	2.1
	$I_0 = \frac{\pi}{2}$	B1	1.1b
	Required area is $2(I_2 - I_{10}) =$ **or** $8\times\frac{1}{4}(I_2 - I_{10}) =$	M1	3.1b
	$= 2\left(\frac{\pi}{4} - \frac{63\pi}{512}\right) = \frac{65\pi}{256}\mathrm{m}^2$	A1	1.1b
		(5)	
			(9 marks)

Notes:

(a)

M1: Splits the integrand into the product shown and begins process of integration by parts (there may be sign errors)

A1: Correct work

M1: Uses limits on the first term and expresses $\cos^2$ term in terms of $\sin^2$

A1*: Completes the proof collecting I_n terms correctly with all stages shown

(b)

M1: Attempts to find I_{10} and/or I_2

M1: Finds I_{10} in terms of I_0

B1: Finds I_0 correctly

M1: States the expression needed to find the required area

A1: Completes the calculation to give this exact answer

Question	Scheme	Marks	AOs
8(a)	$u_1 = 1$ as there is only one way to go up one step	B1	2.4
	$u_2 = 2$ as there are two ways: one step then one step or two steps	B1	2.4
	If first move is one step then can climb the other $(n-1)$ steps in u_{n-1} ways. If first move is two steps can climb the other $(n-2)$ steps in u_{n-2} ways so $u_n = u_{n-1} + u_{n-2}$	B1	2.4
		(3)	
(b)	Sequence begins 1, 2, 3, 5, 8, 13, 21, 34,… so 34 ways of climbing 8 steps	B1	1.1b
		(1)	
(c)	To find general term use $u_n = u_{n-1} + u_{n-2}$ gives $\lambda^2 = \lambda + 1$	M1	2.1
	This has roots $\frac{1 \pm \sqrt{5}}{2}$	A1	1.1b
	So general form is $A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$	M1	2.2a
	Uses initial conditions to find *A* and *B* reaching two equations in *A* and *B*	M1	1.1b
	Obtains $A = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)$ and $B = -\left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)$ and so when $n = 400$ obtains $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{401} - \left(\frac{1-\sqrt{5}}{2}\right)^{401}\right]$ *	A1*	1.1b
		(5)	
		(9 marks)	

Notes:

(a)

B1: Need to see explanation for $u_1 = 1$

B1: Need to see explanation for $u_2 = 2$ with the two ways spelled out

B1: Need to see the first move can be one step or can be two steps and clear explanation of the iterative expression as in the scheme

(b)

B1: The answer is enough for this mark

(c)

M1: Obtains this characteristic equation

A1: Solves quadratic – giving exact answers

M1: Obtains a general form

M1: Use initial conditions to obtains two equations which should be $A\left(1+\sqrt{5}\right) + B\left(1-\sqrt{5}\right) = 2$ o.e. and $A\left(3+\sqrt{5}\right) + B\left(3-\sqrt{5}\right) = 4$ but allow slips here

A1*: Must see exact correct values for *A* and *B* and conclusion given for $n = 400$

Write your name here

Surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number ☐☐☐☐☐
Candidate Number ☐☐☐☐

Further Mathematics

Advanced
Further Mathematics Option 1
Paper 3: Further Statistics 1
Further Mathematics Option 2
Paper 4: Further Statistics 1

Sample Assessment Material for first teaching September 2017
Time: 1 hour 30 minutes

Paper Reference
9FM0/3B
9FM0/4B

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet ‘Mathematical Formulae and Statistical Tables’ is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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S 5 4 4 4 1 A 0 1 2 3

Pearson

Answer ALL questions. Write your answers in the spaces provided.

1. Bacteria are randomly distributed in a river at a rate of 5 per litre of water. A new factory opens and a scientist claims it is polluting the river with bacteria. He takes a sample of 0.5 litres of water from the river near the factory and finds that it contains 7 bacteria. Stating your hypotheses clearly test, at the 5% level of significance, whether there is evidence that the level of pollution has increased.

(5)

Question 1 continued

(Total for Question 1 is 5 marks)

2. A call centre routes incoming telephone calls to agents who have specialist knowledge to deal with the call. The probability of a caller, chosen at random, being connected to the wrong agent is p.

The probability of at least 1 call in 5 consecutive calls being connected to the wrong agent is 0.049

The call centre receives 1000 calls each day.

(a) Find the mean and variance of the number of wrongly connected calls a day.

(7)

(b) Use a Poisson approximation to find, to 3 decimal places, the probability that more than 6 calls each day are connected to the wrong agent.

(2)

(c) Explain why the approximation used in part (b) is valid.

(2)

The probability that more than 6 calls each day are connected to the wrong agent using the binomial distribution is 0.8711 to 4 decimal places.

(d) Comment on the accuracy of your answer in part (b).

(1)

Question 2 continued

Question 2 continued

Question 2 continued

(Total for Question 2 is 12 marks)

3. Bags of £1 coins are paid into a bank. Each bag contains 20 coins.

The bank manager believes that 5% of the £1 coins paid into the bank are fakes.
He decides to use the distribution $X \sim \mathrm{B}(20, 0.05)$ to model the random variable X, the number of fake £1 coins in each bag.

The bank manager checks a random sample of 150 bags of £1 coins and records the number of fake coins found in each bag. His results are summarised in Table 1. He then calculates some of the expected frequencies, correct to 1 decimal place.

Number of fake coins in each bag	0	1	2	3	4 or more
Observed frequency	43	62	26	13	6
Expected frequency	53.8	56.6		8.9	

Table 1

(a) Carry out a hypothesis test, at the 5% significance level, to see if the data supports the bank manager's statistical model. State your hypotheses clearly.

(10)

The assistant manager thinks that a binomial distribution is a good model but suggests that the proportion of fake coins is higher than 5%. She calculates the actual proportion of fake coins in the sample and uses this value to carry out a new hypothesis test on the data. Her expected frequencies are shown in Table 2.

Number of fake coins in each bag	0	1	2	3	4 or more
Observed frequency	43	62	26	13	6
Expected frequency	44.5	55.7	33.2	12.5	4.1

Table 2

(b) Explain why there are 2 degrees of freedom in this case.

(2)

(c) Given that she obtains a χ^2 test statistic of 2.67, test the assistant manager's hypothesis that the binomial distribution is a good model for the number of fake coins in each bag. Use a 5% level of significance and state your hypotheses clearly.

(2)

Question 3 continued

Question 3 continued

Question 3 continued

(Total for Question 3 is 14 marks)

4. A random sample of 100 observations is taken from a Poisson distribution with mean 2.3

Estimate the probability that the mean of the sample is greater than 2.5

(4)

Question 4 continued

(Total for Question 4 is 4 marks)

5. The probability of Richard winning a prize in a game at the fair is 0.15

Richard plays a number of games.

(a) Find the probability of Richard winning his second prize on his 8th game,

(2)

(b) State two assumptions that have to be made, for the model used in part (a) to be valid.

(2)

Mary plays the same game, but has a different probability of winning a prize. She plays until she has won r prizes. The random variable G represents the total number of games Mary plays.

(c) Given that the mean and standard deviation of G are 18 and 6 respectively, determine whether Richard or Mary has the greater probability of winning a prize in a game.

(4)

Question 5 continued

(Total for Question 5 is 8 marks)

6. The probability generating function of the discrete random variable X is given by

$$G_X(t) = k(3 + t + 2t^2)^2$$

(a) Show that $k = \frac{1}{36}$

(2)

(b) Find P(X = 3)

(2)

(c) Show that Var(X) = $\frac{29}{18}$

(8)

(d) Find the probability generating function of $2X + 1$

(2)

Question 6 continued

Question 6 continued

Question 6 continued

(Total for Question 6 is 14 marks)

7. Sam and Tessa are testing a spinner to see if the probability, p, of it landing on red is less than $\frac{1}{5}$. They both use a 10% significance level.

Sam decides to spin the spinner 20 times and record the number of times it lands on red.

(a) Find the critical region for Sam's test.

(2)

(b) Write down the size of Sam's test.

(1)

Tessa decides to spin the spinner until it lands on red and she records the number of spins.

(c) Find the critical region for Tessa's test.

(6)

(d) Find the size of Tessa's test.

(1)

(e) (i) Show that the power function for Sam's test is given by

$$(1 - p)^{19}(1 + 19p)$$

(ii) Find the power function for Tessa's test.

(4)

(f) With reference to parts (b), (d) and (e), state, giving your reasons, whether you would recommend Sam's test or Tessa's test when $p = 0.15$

(4)

Question 7 continued

Question 7 continued

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 7 continued

(Total for Question 7 is 18 marks)

TOTAL FOR PAPER IS 75 MARKS

Paper 3B/4B: Further Statistics 1 Mark Schemes

<table>
<tr><th>Question</th><th colspan="2">Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td rowspan="6">1</td><td colspan="2">$H_0 : \lambda = 5$ $(\lambda = 2.5)$ $H_1 : \lambda > 5$ $(\lambda > 2.5)$</td><td>B1</td><td>2.5</td></tr>
<tr><td colspan="2">$X \sim \text{Po}(2.5)$</td><td>B1</td><td>3.3</td></tr>
<tr><td>Method 1:</td><td>Method 2:</td><td></td><td></td></tr>
<tr><td>$\text{P}(X \geqslant 7) = 1 - \text{P}(X \leqslant 6)$
$= 1 - 0.9858$</td><td>$\text{P}(X \geqslant 5) = 0.1088$
$\text{P}(X \geqslant 6) = 0.042$</td><td>M1</td><td>1.1b</td></tr>
<tr><td>$= 0.0142$</td><td>CR $X \geqslant 6$</td><td>A1</td><td>1.1b</td></tr>
<tr><td colspan="2">$0.0142 < 0.05$ $7 \geqslant 6$ or 7 is in critical region or 7 is significant
Reject H_0. There is evidence at the 5% significance level that the level of pollution has increased.
or
There is evidence to support the scientists claim is justified</td><td>A1cso</td><td>2.2b</td></tr>
<tr><td colspan="5" align="right">(5 marks)</td></tr>
<tr><td colspan="5">Notes:</td></tr>
<tr><td colspan="5">B1: Both hypotheses correct using λ or μ and 5 or 2.5
B1: Realising that the model Po(2.5) is to be used. This may be stated or used
M1: Using or writing $1 - \text{P}(X \leqslant 6)$ or $1 - \text{P}(X < 7)$
a correct CR or $\text{P}(X \geqslant 5)$ = awrt 0.109 and $\text{P}(X \geqslant 6)$ = awrt 0.042
A1: awrt 0.0142 or CR $X \geqslant 6$ or $X > 5$
M1: A fully correct solution and drawing a correct inference in context</td></tr>
</table>

Question	Scheme	Marks	AOs
2(a)	$P(X \geqslant 1) = 1 - P(X = 0)$ $1 - P(X = 0) = 0.049$	B1	3.1b
	$P(X = 0) = 0.951$	B1	1.1b
	$x^5 = 0.951$ $x = 0.99$	M1	3.1b
	$p = 0.01$	A1	1.1b
	$X \sim B(1000, 0.01)$	M1	3.3
	Mean = $np = 10$	A1ft	1.1b
	Variance = $np(1 - p) = 9.9$	A1ft	1.1b
		(7)	
(b)	$X \sim$ Po(“10”) then require: $P(X > 6) = 1 - P(X \leqslant 6)$	M1	3.4
	$= 1 - 0.1301$		
	$= 0.870$	A1	1.1b
		(2)	
(c)	The approximation is valid as : the number of calls is large	B1	2.4
	The probability of connecting to the wrong agent is small	B1	2.4
		(2)	
(d)	The answer is accurate to 2 decimal place	**B1**	3.2b
		(1)	
		(12 marks)	

Notes:

(a)

B1: Realising that the P(at least 1 call) = $1 - P(X = 0)$

B1: Calculating $P(X = 0) = 0.951$

M1: Forming the equation $x^5 = \text{"their 0.951"}$ may be implied by $p = 0.01$

A1: 0.01 only

M1: Realising the need to use the model B(1000, 0.01) This may be stated or used

A1: Mean =10 or ft their p but only if $0 < p < 1$

A1: Var = 9.9 or ft their p but only if $0 < p < 1$

(b)

M1: Using the model Po(“their 10”) (this may be written or used) and $1 - P(X \leqslant 6)$

A1: awrt 0.870 Award M1 A1 for awrt 0.870 with no incorrect working

(c)

B1: Explaining why approximation is valid - need the context of number and calls

B1: Need the context connecting, wrong agent

(d)

B1: Evaluating the accuracy of their answer in (b). Allow 2 significant figures

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td rowspan="11">3(a)</td><td>Expected value for 2 = $150 \times \mathrm{P}(X = 2)$</td><td>M1</td><td>3.4</td></tr>
<tr><td>= 28.3015…</td><td>A1</td><td>1.1b</td></tr>
<tr><td>Expected value for 4 or more = 150 – (53.8 + 56.6 + 28.3 + 8.9)
= 2.4</td><td>A1ft</td><td>1.1b</td></tr>
<tr><td>H_0: Bin(20, 0.05) is a suitable model
H_1: Bin(20, 0.05) is not a suitable model</td><td>B1</td><td>2.5</td></tr>
<tr><td>Combining last two groups<table><tr><td></td><td>$\geqslant$ 3</td></tr><tr><td>Observed frequency</td><td>19</td></tr><tr><td>Expected frequency</td><td>11.3</td></tr></table></td><td>M1</td><td>2.1</td></tr>
<tr><td>$\nu = 4 - 1 = 3$</td><td>B1</td><td>1.1b</td></tr>
<tr><td>Critical value, χ^2 (0.05) = 7.815</td><td>B1</td><td>1.1a</td></tr>
<tr><td>Test statistic = $\frac{(43-53.8)^2}{53.8} + \frac{(62-56.6)^2}{56.6} + \ldots$</td><td>M1</td><td>1.1b</td></tr>
<tr><td>= 8.117</td><td>A1</td><td>1.1b</td></tr>
<tr><td>In critical region, sufficient evidence to reject H_0, accept H_1
Significant evidence at 5% level to reject the manager’s model</td><td>A1</td><td>3.5a</td></tr>
<tr><td></td><td>(10)</td><td></td></tr>
<tr><td rowspan="4">(b)</td><td>$\nu = 4 - 2 = 2$</td><td></td><td></td></tr>
<tr><td>4 classes due to pooling</td><td>B1</td><td>2.4</td></tr>
<tr><td>2 restrictions (equal total and mean/proportion)</td><td>B1</td><td>2.4</td></tr>
<tr><td></td><td>(2)</td><td></td></tr>
<tr><td rowspan="3">(c)</td><td>H_0: Binomial distribution is a good model
H_1: Binomial distribution is not a good model</td><td>B1</td><td>3.4</td></tr>
<tr><td>Critical value, χ^2 (0.05) = 5.991
Test statistic is not in critical region, insufficient evidence to reject H_0
There is evidence that the Binomial distribution is a good model</td><td>B1</td><td>3.5a</td></tr>
<tr><td></td><td>(2)</td><td></td></tr>
<tr><td colspan="4" align="right">(14 marks)</td></tr>
</table>

Question 3 notes:

(a)

M1: Using the binomial model $150 \times p^2 \times (1 - p)^{18}$ may be implied by 28.3

A1: awrt 28.3

A1: awrt 2.4 or ft their "28.3"

B1: Both hypotheses correct using the correct notation or written out in full

M1: For recognising the need to combine groups

B1: Number of degrees of freedom = 3 may be implied by a correct CV

B1: awrt 7.82

M1: Attempting to find $\sum \frac{(O_i - E_i)^2}{E_i}$ or $\sum \frac{O_i^2}{E_i} - N$ may be implied by awrt 8.12

A1: awrt 8.12

A1: Evaluating the outcome of a model by drawing a correct inference in context

(b)

B1: Explaining why there are 4 classes

B1: Explanation of why 2 is subtracted

(c)

B1: Correct hypotheses for the refined model

B1: The CV awrt 5.99 and drawing the correct inference for the refined model

Question	Scheme	Marks	AOs
4	Po(2.3) $n = 100$ $\mu = 2.3$ $\sigma^2 = 2.3$		
	$\text{CLT} \Rightarrow \bar{X} \approx \text{N}\left(2.3, \frac{2.3}{100}\right)$	M1 A1	3.1a 1.1b
	$\text{P}(\bar{X} > 2.5) = \text{P}\left(Z > \frac{2.5 - 2.3}{\sqrt{0.023}}\right)$	M1	3.4
	$= \text{P}(\text{Z} > 1.318..)$		
	$= 0.09632\ldots$	A1	1.1b
		(4)	
		(4 marks)	

Notes:

M1: For realising the need to use the CLT to set $\bar{X} \approx$ normal with correct mean
May be implied by using the correct normal distribution

A1: For fully correct normal stated or used

M1: Use of the normal model to find $\text{P}(\bar{X} > 2.5)$. Can be awarded for $\frac{2.5-2.3}{\sqrt{0.023}}$
or awrt 1.32

A1: awrt 0.0963

Question	Scheme	Marks	AOs
5(a)	$\binom{7}{1} \times 0.15^2 \times (0.85)^6$	M1	3.3
	= 0.05940… = awrt **<u>0.0594</u>**	A1	1.1b
		(2)	
(b)	The model is only valid if:		
	the games (trials) are **independent**	B1	3.5b
	the probability of winning a prize, 0.15, is **constant** for each game	B1	3.5b
		(2)	
(c)	$18 = \frac{r}{p}$ and $6^2 = \frac{r(1-p)}{p^2}$	M1 A1	3.1b 1.1b
	Solving: $2p = 1 - p$	M1	1.1b
	$p = \frac{1}{3}$ (> 0.15) so Mary has the greater chance of winning a prize	A1	3.2a
		(4)	
	(8 marks)		

Notes:

5(a)
M1: For selecting an appropriate model negative binomial or B(7, 0.15) with an extra success in 8th trial e.g.
$\binom{7}{1} 0.15 \times (0.85)^6 \times 0.15$ Allow $\binom{7}{1} 0.85 \times (0.15)^6 \times 0.85$ may be implied by awrt 0.0594
A1: awrt 0.0594

(b)
B1: Stating the first assumption that games are independent
B1: Stating the second assumption that the probability remains constant

(c)
M1: Forming an equation for the mean or for the standard deviation
A1: Both equations correct
M1: Solving the 2 equations leading to $2p = 1 - p$
A1: For $p = \frac{1}{3}$ followed by a correct deduction

Question	Scheme	Marks	AOs
6(a)	$\mathrm{G}_X(1)=1$ gives	M1	2.1
	$k\times 6^2=1$ so $k=\frac{1}{36}$ *	A1*cso	1.1b
		(2)	
(b)	P(X = 3) = coefficient of t^3 so $\mathrm{G}_X(t)=k\left(...+4t^3...\right)$	M1	1.1b
	[P(X = 3) =] $\underline{\frac{1}{9}}$	A1	1.1b
		(2)	
(c)	$\mathrm{G}'_X(t)=2k\left(3+t+2t^2\right)\times\left(1+4t\right)$	M1	2.1
	E(X) = $\mathrm{G}'_X(1)=2k\left(3+1+2\right)\times\left(1+4\right)$	M1	1.1b
	$=\frac{5}{3}$	A1	1.1b
	$\mathrm{G}''_X(t)=2k\left[\left(3+t+2t^2\right)\times 4+\left(1+4t\right)^2\right]$	M1 A1	2.1 1.1b
	$\mathrm{G}''_X(1)=2k[6\times 4+5^2]$ $\left\{=\frac{49}{18}\right\}$	M1	1.1b
	Var(X) = $\mathrm{G}''_X(1)+\mathrm{G}'_X(1)-\left[\mathrm{G}'_X(1)\right]^2=\frac{49}{18}+\frac{5}{3}-\frac{25}{9}$	M1	2.1
	$=\underline{\frac{29}{18}}$ *	A1*cso	1.1b
		(8)	
(d)	$\mathrm{G}_{2X+1}(t)=\frac{t}{36}\left(3+t^2+2\left(t^2\right)^2\right)^2$ [×t <u>or</u> sub t^2 for t]	M1	3.1a
	$=\mathrm{G}_{2X+1}(t)=\frac{t}{36}\left(3+t^2+2t^4\right)^2$	A1	1.1b
		(2)	
			(14 marks)

Notes:

(a)

M1: Stating $\mathrm{G}_X(1)=1$

A1*: Fully correct proof with no errors cso

(b)

M1: Attempting to find the coefficient of t^3. May be implied by obtaining $\frac{1}{9}$ or awrt 0.11

A1: $\frac{1}{9}$, allow awrt 0.111

Question 6 notes continued:	
(c)	
M1:	Attempting to find $G'_X(t)$. Allow Chain rule or multiplying out the brackets and differentiating
M1:	Substituting t = 1 into $G'_X(t)$
A1:	$\frac{5}{3}$, allow awrt 1.67
M1:	Attempting to find $G''_X(t)$
A1:	$2k\left[\left(3+t+2t^2\right)\times 4+\left(1+4t\right)^2\right]$ **or** $k(48t^2+24t+26)$ o.e.
A1:	$2k[6\times 4+5^2]$ o.e.
M1:	Using $G''_X(1)+G'_X(1)-\left[G'_X(1)\right]^2$ to find the Variance
A1*:	$\frac{29}{18}$ cso
(d)	
M1:	Realising the need to $\times t$ **or** sub t^2 for t
A1:	$\frac{t}{36}\left(3+t^2+2t^4\right)^2$, or $\frac{t}{36}\left(9+6t^2+13t^4+4t^6+4t^8\right)$ o.e.

Question	Scheme	Marks	AOs
7(a)	$X \sim \mathrm{B}(20, 0.2)$ and seek c such that $\mathrm{P}(X \leqslant c) < 0.10$	M1	3.3
	$[\mathrm{P}(X \leqslant 1) = 0.0692]$ CR is $X \leqslant 1$	A1	1.1b
		(2)	
(b)	Size = **<u>0.0692</u>**	B1ft	1.2
		(1)	
(c)	Y = no. of spins until red obtained so $Y \sim \mathrm{Geo}(0.2)$	M1	3.3
	$\mu = \frac{1}{p}$ so if $p < 0.2$ then mean is <u>larger</u> so seek d so that $\mathrm{P}(Y \geqslant d) < 0.10$	M1	2.4
	$\mathrm{P}(Y \geqslant d) = (0.8)^{d-1}$	M1	3.4
	$(0.8)^{d-1} < 0.10 \quad \Rightarrow \quad d - 1 > \frac{\log(0.1)}{\log(0.8)}$	M1	1.1b
	$d > 11.3..$	A1	1.1b
	CR is $\boldsymbol{Y \geqslant 12}$	A1	2.2b
		(6)	
(d)	Size = $[0.8^{11} = 0.085899\ldots]$ = **<u>0.0859</u>**	B1	1.1b
		(1)	
(e)(i)	Power = P(reject H0 when it is false) = $\mathrm{P}(X \leqslant 1 \mid X \sim \mathrm{B}(20, p))$	M1	2.1
	$= (1-p)^{20} + 20(1-p)^{19} p$	M1	1.1b
	$= (1-p)^{19}(1+19p)$ *	A1*cso	1.1b
(ii)	Power = $(1-p)^{11}$	B1	1.1b
		(4)	
(f)	Sam's test has smaller P(Type I error) (or size) so is better	B1	2.2a
	Power of Sam's test = 0.1755…	B1	1.1b
	Power of Tessa's test = 0.85^{11} = 0.1673…	B1	1.1b
	So for $p = 0.15$ **Sam's test** is recommended	B1	2.2b
		(4)	
			(18 marks)

Question 7 notes:	
(a)	
M1:	Realising the need to use the model Using B(20,0.2) with method for finding the CR or implied by a correct CR
A1:	$X \leqslant 1$ or $X < 2$
(b)	
B1:	awrt 0.0692
(c)	
M1:	Realising that the model Geo(0.2)is needed. This may be written or used
M1:	Realising the key step that they need to find $P(Y \geqslant d) < 0.10$
M1:	**U**sing the model $(0.8)^{d-1}$
M1:	Using the model $(0.8)^{d-1} < 0.10$ and finding a method to solve leading to a value/range of values for d
A1:	For $d > 11.3..$
A1:	For $Y \geqslant 12$ or $Y > 11$ (a correct inference)
(d)	
B1ft:	awrt 0.0692. ft their answer to part (c)
(e)(i)	
M1:	Using B(20, p) and realizing they need to find $P(X \leqslant 1)$ o.e. This may be used or written
M1:	Using $P(X = 0) + P(X = 1)$
A1*:	Fully correct proof (no errors) cso
(ii)	
B1:	For $(1-p)^{11}$
(f)	
B1:	Making a deduction about the tests using the answers to part(b) and (d)
B1:	awrt 0.0176
B1:	awrt 0.167
B1:	A correct inference about which test is recommended

Write your name here

Surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number ☐☐☐☐☐

Candidate Number ☐☐☐☐

Further Mathematics

Advanced
Further Mathematics Option 2
Paper 4: Further Statistics 2

Sample Assessment Material for first teaching September 2017
Time: 1 hour 30 minutes

Paper Reference
9FM0/4E

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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Pearson

Answer ALL questions. Write your answers in the spaces provided.

1. The three independent random variables A, B and C each have a continuous uniform distribution over the interval [0, 5].

(a) Find the probability that A, B and C are all greater than 3

(3)

The random variable Y represents the maximum value of A, B and C.

The cumulative distribution function of Y is

$$\mathrm{F}(y) = \begin{cases} 0 & y < 0 \\ \dfrac{y^3}{125} & 0 \leqslant y \leqslant 5 \\ 1 & y > 5 \end{cases}$$

(b) Using algebraic integration, show that Var $(Y) = 0.9375$

(4)

(c) Find the mode of Y, giving a reason for your answer.

(2)

(d) Describe the skewness of the distribution of Y. Give a reason for your answer.

(1)

(e) Find the value of k such that $\mathrm{P}(k < Y < 2k) = 0.189$

(3)

Question 1 continued

Question 1 continued

Question 1 continued

(Total for Question 1 is 13 marks)

2. A researcher claims that, at a river bend, the water gradually gets deeper as the distance from the inner bank increases. He measures the distance from the inner bank, b cm, and the depth of a river, s cm, at 7 positions. The results are shown in the table below.

Position	A	B	C	D	E	F	G
Distance from inner bank b cm	100	200	300	400	500	600	700
Depth s cm	60	75	85	76	110	120	104

The Spearman's rank correlation coefficient between b and s is $\frac{6}{7}$

(a) Stating your hypotheses clearly, test whether or not the data provides support for the researcher's claim. Use a 1% level of significance.

(4)

(b) Without re-calculating the correlation coefficient, explain how the Spearman's rank correlation coefficient would change if

(i) the depth for G is 109 instead of 104

(ii) an extra value H with distance from the inner bank of 800 cm and depth 130 cm is included.

(3)

The researcher decided to collect extra data and found that there were now many tied ranks.

(c) Describe how you would find the correlation with many tied ranks.

(2)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 2 continued

(Total for Question 2 is 9 marks)

3. A nutritionist studied the levels of cholesterol, X mg/cm^3, of male students at a large college. She assumed that X was distributed $\mathrm{N}(\mu, \sigma^2)$ and examined a random sample of 25 male students. Using this sample she obtained unbiased estimates of μ and σ^2 as $\hat{\mu}$ and $\hat{\sigma}^2$

A 95% confidence interval for μ was found to be (1.128, 2.232)

(a) Show that $\hat{\sigma}^2 = 1.79$ (correct to 3 significant figures)

(4)

(b) Obtain a 95% confidence interval for σ^2

(3)

Question 3 continued

(Total for Question 3 is 7 marks)

4. The times, x seconds, taken by the competitors in the 100 m freestyle events at a school swimming gala are recorded. The following statistics are obtained from the data.

	No. of competitors	**Sample mean $\bar{x}$**	$\sum x^2$
Girls	8	83.1	55 746
Boys	7	88.9	56 130

Following the gala, a mother claims that girls are faster swimmers than boys. Assuming that the times taken by the competitors are two independent random samples from normal distributions,

(a) test, at the 10% level of significance, whether or not the variances of the two distributions are the same. State your hypotheses clearly.

(7)

(b) Stating your hypotheses clearly, test the mother's claim. Use a 5% level of significance.

(6)

Question 4 continued

Question 4 continued

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 4 continued

(Total for Question 4 is 13 marks)

5. Scaffolding poles come in two sizes, long and short. The length L of a long pole has the normal distribution $N(19.6, 0.6^2)$. The length S of a short pole has the normal distribution $N(4.8, 0.3^2)$. The random variables L and S are independent.

A long pole and a short pole are selected at random.

(a) Find the probability that the length of the long pole is more than 4 times the length of the short pole. Show your working clearly.

(6)

Four short poles are selected at random and placed end to end in a row. The random variable T represents the length of the row.

(b) Find the distribution of T.

(3)

(c) Find $P(|L - T| < 0.2)$

(4)

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 5 continued

Question 5 continued

Question 5 continued

(Total for Question 5 is 13 marks)

6. A random sample of 10 female pigs was taken. The number of piglets, x, born to each female pig and their average weight at birth, m kg, was recorded. The results were as follows:

Number of piglets, x	4	5	6	7	8	9	10	11	12	13
Average weight at birth, m kg	1.50	1.20	1.40	1.40	1.23	1.30	1.20	1.15	1.25	1.15

(You may use $S_{xx} = 82.5$ and $S_{mm} = 0.12756$ and $S_{xm} = -2.29$)

(a) Find the equation of the regression line of m on x in the form $m = a + bx$ as a model for these results.
(2)

(b) Show that the residual sum of squares (RSS) is 0.064 to 3 decimal places.
(2)

(c) Calculate the residual values.
(2)

(d) Write down the outlier.
(1)

(e) (i) Comment on the validity of ignoring this outlier.

(ii) Ignoring the outlier, produce another model.

(iii) Use this model to estimate the average weight at birth if $x = 15$

(iv) Comment, giving a reason, on the reliability of your estimate.
(5)

Question 6 continued

Question 6 continued

Question 6 continued

(Total for Question 6 is 12 marks)

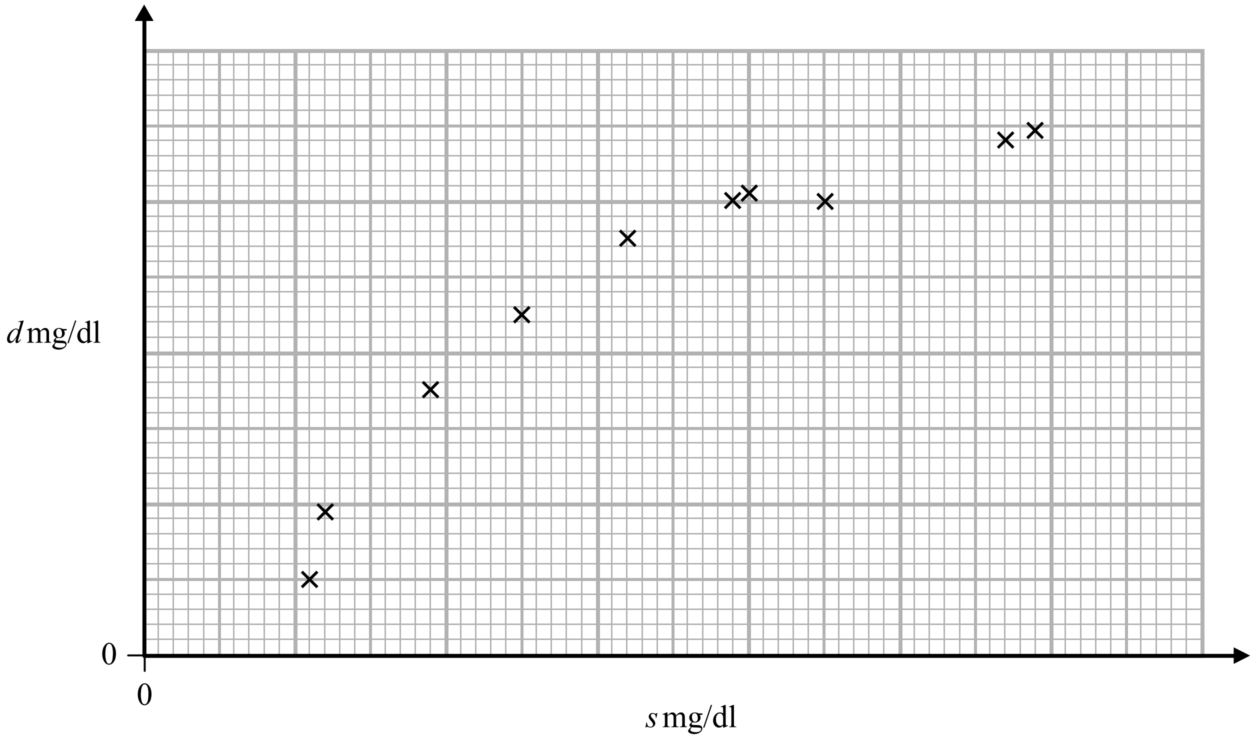
7. Over a period of time, researchers took 10 blood samples from one patient with a blood disease. For each sample, they measured the levels of serum magnesium, s mg/dl, in the blood and the corresponding level of the disease protein, d mg/dl. One of the researchers coded the data for each sample using $x = 10s$ and $y = 10(d - 9)$ but spilt ink over his work.

The following summary statistics and unfinished scatter diagram are the only remaining information.

$$\sum d^2 = 1081.74 \qquad S_{ds} = 59.524$$

and

$$\sum y = 64 \qquad S_{xx} = 2658.9$$



(a) Use the formula for S_{xx} to show that $S_{ss} = 26.589$

(3)

(b) Find the value of the product moment correlation coefficient between s and d.

(4)

(c) With reference to the unfinished scatter diagram, comment on your result in part (b).

(1)

Question 7 continued

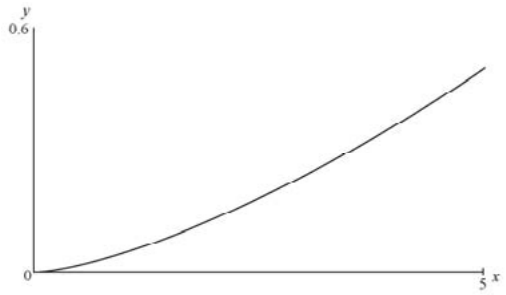
Question 7 continued

Question 7 continued

(Total for Question 7 is 8 marks)

TOTAL FOR PAPER IS 75 MARKS

Paper 4E: Further Statistics 2 Mark Scheme

Question	Scheme	Marks	AOs
1(a)	$\mathrm{P}(A > 3) = \frac{2}{5}$	B1	1.1b
	$\left(\frac{2}{5}\right)^3 = \frac{8}{125}$	M1 A1	1.1a 1.1b
		(3)	
(b)	$\mathrm{f}(y) = \frac{3y^2}{125}$	M1	2.1
	$\mathrm{E}(Y) = \int_0^5 \frac{3y^3}{125}\,\mathrm{d}y$ $= \left[\frac{3y^4}{500}\right]_0^5 \qquad \left[= \frac{15}{4}\right]$	M1	1.1b
	$\mathrm{Var}(Y) = \int_0^5 \left(\frac{3y^4}{125}\right)\mathrm{d}y - \left(\frac{15}{4}\right)^2$	M1	1.1b
	$= 0.9375^*$	A1*cso	1.1b
		(4)	
(c)	Mode = 5	B1	1.2
	 Or reason based on $\frac{\mathrm{df}(y)}{\mathrm{d}y} > 0$	B1	2.4
		(2)	
(d)	From a sketch or mode > mean therefore it has negative skew	B1ft	2.4
		(1)	
(e)	$\frac{(2k)^3}{125} - \frac{k^3}{125} = 0.189$	M1	3.1a
	$\frac{7k^3}{125} = 0.189$	A1	1.1b
	$k = 1.5$	A1	1.1b
		(3)	
		(13 marks)	

Question 1 notes:

(a)

B1: $\frac{2}{5}$ o.e. may be implied by a correct answer

M1: $\left(\text{"their}\left(\frac{2}{5}\right)\text{"}\right)^3$ may be implied by a correct answer

A1: $\frac{8}{125}$ o.e.

(b)

M1: Realising that firstly need to find pdf f(y) and attempt to differentiate F(y)

M1: Continuing the argument with an attempt to integrate $y \times$ "their f(y)"
$y^n \to y^{n+1}$

M1: Integrating $y^2 \times$ "their f(y)" - ["their E(Y)"]^2 $y^n \to y^{n+1}$

A1*: Complete correct solution no errors

(c)

B1: 5 only

B1: Explain their reason by either an accurate sketch or $\frac{\mathrm{df}(y)}{\mathrm{d}y} > 0$ therefore an increasing function o.e.

(d)

B1ft: Explaining the reason for their answer. Follow through their part(b) or mean from(d) and mode from(c). A correct sketch of "their f(y)" – may be seen anywhere in question or ft their mean and mode plus a correct conclusion

NB: Watch for gaming. A student who writes both negative skew with a reason and positive skew with a reason. Please send these to your Team Leader

(e)

M1: Attempting to translate the problem into an equation using 2k and k. Allow if the brackets are missing e.g. $\frac{2k^3}{125} - \frac{k^3}{125}$. No need for the 0.189

A1: A correct equation in any form

A1: A correct answer only

Question	Scheme	Marks	AOs
2(a)	$\mathrm{H}_0 : \rho = 0, \mathrm{H}_1 : \rho > 0$	B1	2.5
	Critical value at 1% level is 0.8929	B1	1.1b
	$r_s < 0.8929$ so not significant evidence to reject H_0	M1	2.1
	The researcher's claim is not correct (at 1% level) **or** insufficient evidence for researcher's claim **or** there is insufficient evidence that water gets deeper further from inner bank **or** no (positive) correlation between depth of water and distance from inner bank	A1ft	2.2b
		(4)	
(b)(i)	The **ranks will remain the same** therefore there will be **no change** to the spearman's rank correlation coefficient	**B1**	2.4
(ii)	Spearman's rank correlation coefficient will **increase** since	**B1**	2.2a
	The **ranks are the same** for both distance and depth therefore $\boldsymbol{d} = \mathbf{0}$ however, $\boldsymbol{n}$ **has increased or** the new position follows the pattern that large b is associated with large s and so r_s will increase	**B1**	2.4
		(3)	
(c)	The mean of the tied ranks is given to each…	B1	2.4
	… then use PMCC	B1	2.4
		(2)	
		(9 marks)	

Notes:

(a)
B1: Both hypotheses correct written using the notation ρ
B1: awrt 0.893
M1: Drawing a correct inference using their answer to part(a) and their CV
A1ft: Drawing a correct inference in context using their answer to part(a) and their CV

(b)(i)
B1: Stating **no change** and an explanation including **ranks remain unchanged** o.e. and **no change o.e.**

(b)(ii)
B1: Interpreted the outcome of adding a point as **increased** oe
B1: Explaining why. Need to mention the **ranks are the same for both oe** and $\boldsymbol{n}$ **has increased oe**

(c)
B1: Explaining that the mean of the values for the tied ranks is given to both values
B1: Explaining that the PMCC must be used

Question	Scheme	Marks	AOs
3(a)	95% CI for μ uses t value of **2.064**	B1	3.3
	$\frac{\hat{\sigma}}{\sqrt{25}}\times"2.064"=\frac{1}{2}(2.232-1.128)$ <u>or</u> $\frac{1}{2}(2.232+1.128)+"2.064"\times\frac{\hat{\sigma}}{\sqrt{25}}=2.232$ (oe)	M1	2.1
	$\hat{\sigma}=\frac{2.76}{"2.064"}$ <u>or</u> 1.3372…	M1	1.1b
	$\hat{\sigma}^2=1.788...$ [=1.79 (3sf)] *	A1*cso	1.1b
		(4)	
(b)	12.401, $<\frac{24\times1.79}{\sigma^2}<$, 39.364	B1 M1	1.1b 1.1a
	<u>**1.09**</u> $<\sigma^2<$ <u>**3.46**</u>	A1	1.1b
		(3)	
		(7 marks)	

Notes:

(a)

B1: Realising that the *t*-distribution must be used as a model and finding the correct value awrt 2.06

M1: Using the correct formula with a *t*-value, $\frac{\hat{\sigma}}{\sqrt{25}}\times"t\text{ value}"=\frac{1}{2}(2.232-1.128)$ or

$$\frac{1}{2}(2.232+1.128)+"t\text{ value}"\times\frac{\hat{\sigma}}{\sqrt{25}}=2.232 \quad \textbf{or}$$

$$\frac{1}{2}(2.232+1.128)-"t\text{ value}"\times\frac{\hat{\sigma}}{\sqrt{25}}=1.128$$

M1: Rearranging one of these formula accurately to find a value of $\hat{\sigma}$

A1cso*: A correct solution only using awrt 1.79

(b)

B1: awrt 12.4 or 39.4 May be implied by a correct confidence interval

M1: $\frac{24\times1.79}{\sigma^2}$ May be implied by a correct confidence interval

A1: awrt 1.09 and awrt 3.46

Question	Scheme	Marks	AOs
4(a)	H$_0$: $\sigma_G^2 = \sigma_B^2$, H$_1$: $\sigma_G^2 \neq \sigma_B^2$,	B1	2.5
	$s_B^2 = \frac{1}{6}(56130 - 7 \times 88.9^2) = \dfrac{807.53}{6} = 134.6$	M1 A1	2.1 1.1b
	$s_G^2 = \frac{1}{7}(55746 - 8 \times 83.1^2) = \dfrac{501.12}{7} = 71.58$	A1	1.1b
	$\dfrac{s_B^2}{s_G^2} = 1.880\ldots$	M1	3.4
	Critical value $F_{6,\,7} = 3.87$	B1	1.1b
	Not significant, variances can be treated as the same	A1 ft	2.2b
		(7)	
(b)	H$_0$: $\mu_B = \mu_G$, H$_1$: $\mu_B > \mu_G$	B1	2.5
	Pooled estimate of variance $s^2 = \dfrac{6\times134.6+7\times71.58}{13} = 100.6653\ldots$	M1	3.1b
	Test statistic $t = \dfrac{88.9 - 83.1}{s\sqrt{\frac{1}{7}+\frac{1}{8}}}$ = awrt 1.12	M1 A1	1.1b 1.1b
	Critical value $t_{13}(5\%) = 1.771$	B1	1.1b
	Insufficient evidence to support mother's claim	A1 ft	2.2b
		(6)	
		(13 marks)	

Notes:

(a)

B1: Both hypotheses correct using the notation σ^2. Allow σ rather than σ^2

M1: Using a correct Method for either s_B^2 or s_G^2 May be implied by a correct value

A1: awrt 135

A1: awrt 71.6

M1: Using the F-distribution as the model e.g. $\dfrac{s_B^2}{s_G^2}$

B1: awrt 3.87

A1ft: Drawing a correct inference following through their CV and value for $\dfrac{s_B^2}{s_G^2}$

(b)

B1: Both hypotheses correct using the notation μ

M1: For realising the need to find the pooled estimate for the test require from a correct interpretation of the question

M1: Correct method for test statistic $t = \dfrac{88.9-83.1}{\text{"their } s\text{"}\sqrt{\frac{1}{7}+\frac{1}{8}}}$ May be implied by a correct awrt 1.12

A1: awrt 1.12

B1: awrt 1.77

A1ft: Drawing a correct inference following through their CV and test statistic

Question	Scheme	Marks	AOs
5(a)	Let $X = L - 4S$ then E(X) = 19.6 – 4 × 4.8	M1	2.3
	= 0.4	A1	1.1b
	Var(X) = $\text{Var}(L) + 4^2\text{Var}(S) = 0.6^2 + 16 \times 0.3^2$	M1	2.1
	= 1.8	A1	1.1b
	P(X > 0) = [P($Z > \dfrac{0-0.4}{\sqrt{1.8}} = -0.298......$)]	M1	2.1
	= 0.617202... awrt **<u>0.617</u>**	A1	1.1b
		(6)	
(b)	$T = S_1 + S_2 + S_3 + S_4$ (May be implied by 0.36)	M1	3.3
	T ~ N(19.2, 0.36) E(T) = 19.2	B1	1.1b
	Var(T) = 0.36 **or** 0.6^2	A1	1.1b
		(3)	
(c)	Let $Y = L - T$ E(Y) = E(L) – E(T) = [0.4]	M1	3.3
	Var(Y) = Var(L) + Var(T) = [0.72]	M1	1.1b
	Require P(– 0.2 < Y < 0.2)	M1	3.1a
	= 0.16708… awrt **<u>0.167</u>**	A1	1.1b
		(4)	
			(13 marks)

Notes:

(a)

M1: Selecting and using an appropriate model i.e. $\pm(L - 4S)$. May be implied by 0.4

A1: 0.4 oe

M1: For realising the need to use Var(L) + 4^2Var(S). Allow use of 0.6 for Var(L) instead of 0.6^2 and/or 0.3 for Var(S) instead of 0.3^2 may be implied by 1.8

A1: 1.8 only

M1: For realising P(X > 0) is required and an attempt to find it e.g. $\dfrac{0-0.4}{\sqrt{\text{"their Var}(X)\text{"}}}$ but do not allow a negative Var(X)

A1: awrt 0.617

(b)

M1: Selecting and using an appropriate model ie $S_1 + S_2 + S_3 + S_4$: may be implied by 0.36

B1: 19.2 only

A1: 0.36

(c)

M1: Setting up and using the model $Y = L - T$. May be implied by $E(Y) = E(L) - E(T)$

M1: Using Var(Y) = Var(L) + Var(T)

M1: Dealing with the modulus and realising they need to find P(– 0.2 < Y < 0.2)

A1: awrt 0.167

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td rowspan="4">6(a)</td><td>$\left[b = \frac{S_{xm}}{S_{xx}} = -\ 0.0277576 \right]$</td><td>M1</td><td>3.3</td></tr>
<tr><td>$[a = \bar{m} - b\bar{x} = 1.278 + 0.0277576 \times 8.5 = 1.5139]$</td><td></td><td></td></tr>
<tr><td>$m = 1.5139 - 0.02775\ldots x$</td><td>A1</td><td>1.1b</td></tr>
<tr><td></td><td>(2)</td><td></td></tr>
<tr><td rowspan="3">(b)</td><td>$\text{RSS} = 0.12756 - \frac{(-2.29)^2}{82.5}$</td><td>M1</td><td>1.1b</td></tr>
<tr><td>$= 0.06399$*</td><td>A1*</td><td>1.1b</td></tr>
<tr><td></td><td>(2)</td><td></td></tr>
<tr><td rowspan="2">(c)</td><td>
<table>
<tr><td>x</td><td>m</td><td>$m = a + bx$</td><td>ε</td></tr>
<tr><td>4</td><td>1.50</td><td>1.4029</td><td>+0.0971</td></tr>
<tr><td>5</td><td>1.20</td><td>1.3752</td><td>– 0.1752</td></tr>
<tr><td>6</td><td>1.40</td><td>1.3474</td><td>+0.0526</td></tr>
<tr><td>7</td><td>1.40</td><td>1.3196</td><td>+0.0804</td></tr>
<tr><td>8</td><td>1.23</td><td>1.2919</td><td>– 0.0619</td></tr>
<tr><td>9</td><td>1.30</td><td>1.2641</td><td>+0.0359</td></tr>
<tr><td>10</td><td>1.20</td><td>1.2364</td><td>– 0.0364</td></tr>
<tr><td>11</td><td>1.15</td><td>1.2086</td><td>– 0.0586</td></tr>
<tr><td>12</td><td>1.25</td><td>1.1808</td><td>+0.0692</td></tr>
<tr><td>13</td><td>1.15</td><td>1.1531</td><td>– 0.0031</td></tr>
</table>
</td><td>M1
A1</td><td>3.4
1.1b</td></tr>
<tr><td></td><td>(2)</td><td></td></tr>
<tr><td rowspan="2">(d)</td><td>The point (5, 1.2) is an outlier</td><td>B1ft</td><td>2.2b</td></tr>
<tr><td></td><td>(1)</td><td></td></tr>
<tr><td>(e)(i)</td><td>It is a valid piece of data so should be used
or
It does not follow the pattern according to the residuals so may contain an error making the result invalid so should be removed</td><td>B1</td><td>2.4</td></tr>
<tr><td rowspan="2">(ii)</td><td>$a = \bar{m} - b\bar{x} = 1.28667 + 0.03765 \times 8.88889 = 1.6213$</td><td>M1</td><td>3.3</td></tr>
<tr><td>$m = 1.6213 - 0.03765x$</td><td>A1</td><td>1.1b</td></tr>
<tr><td rowspan="2">(iii)</td><td>$m = 1.6213 - 0.03765 \times 15$</td><td></td><td></td></tr>
<tr><td>$= 1.056$ or awrt 1.06</td><td>B1ft</td><td>3.4</td></tr>
<tr><td rowspan="2">(iv)</td><td>The model is only reliable if the values are limited to those in the given range so probably not reliable</td><td>B1</td><td>3.5b</td></tr>
<tr><td></td><td>(5)</td><td></td></tr>
<tr><td colspan="4" align="right">(12 marks)</td></tr>
</table>

Question 6 notes:	
6(a)	
M1:	Realising the need to use $b = \frac{S_{xm}}{S_{xx}}$ and $a = \overline{m} - b\overline{x}$
A1:	m = awrt 1.51) – (awrt 0.0278) x. Award M1A1 for correct equation
(b)	
M1:	Using $S_{mm} - \frac{\left(S_{xm}\right)^2}{S_{xx}}$
A1*:	awrt 0.064
(c)	
M1:	Using the model in part (a) i.e. m – (“1.5139” – “0.02775”x) implied by a correct value
A1:	All correct. Award M1A1 for a list of correct residuals
(d)	
B1:	Inferring from the residuals that the outlier is (5, 1.2) ft their residuals.
(e)(i)	
B1:	Explaining why the outlier should be removed or not.
(ii)	
M1:	Removing the outlier and refining the model by finding a new regression line.
A1:	m = (awrt 1.62) – (awrt 0.0377)x
(iii)	
B1ft:	using their model in e(i) with $x = 15$. awrt 1.06 or ft their e(ii)
(iv)	
B1:	Realising the limitations of the model by stating it is <u>not reliable</u> and giving the reason why i.e. extrapolation/out of range o.e.

Question	Scheme	Marks	AOs
7(a)	$S_{xx} = \sum(10s)^2 - \frac{\left(\sum 10s\right)^2}{10}$	M1	2.1
	$2658.9 = 100\sum(s)^2 - \frac{100\left(\sum s\right)^2}{10}$	M1	1.1b
	$2658.9 = 100\ S_{ss}$		
	$S_{ss} = 26.589$ *	A1*cso	1.1b
		(3)	
(b)	$64 = \sum_{1}^{10} 10(d_i - 9)$	M1	3.1a
	$64 = 10\sum_{1}^{10} d_i - 900$		
	$\sum_{1}^{10} d_i = 96.4$	A1	1.1b
	$S_{dd} = 1081.74 - \frac{(\text{"96.4"})^2}{10}$	M1	1.1b
	$= 152.444$		
	$r = 0.935$	A1ft	1.1b
		(4)	
(c)	Linear correlation is significant but scatter diagram suggests a non-linear relationship between the level of serum magnesium, and the level of the disease protein	B1	3.5a
		(1)	
		(8 marks)	

Notes:

(a)

M1: Attempting to use $S_{xx} = \sum x^2 - \frac{\left(\sum x\right)^2}{10}$ with $x = 10s$

M1: Substituting in 2658.9 and dealing with the 10 correctly

A1*: cso A complete solution with no errors leading to 26.589 only

(b)

M1: Realising that either $64 = \sum_{1}^{10} 10(d_i - 9)$ or $64 = 10\sum_{1}^{10} d_i - 900$ o.e. must be used. May be implied by seeing 96.4

A1: 96.4 only

M1: Attempting to use $S_{dd} = \sum d^2 - \frac{\left(\sum d\right)^2}{10}$ may be implied by 0.935

A1ft: awrt 0.935 ft "their 96.4"

(c)

B1: A correct comment comparing their value of r and the scatter diagram in context

Write your name here

Surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number	Candidate Number

Further Mathematics

Advanced
Further Mathematics Option 1
Paper 3: Further Mechanics 1
Further Mathematics Option 2
Paper 4: Further Mechanics 1

Sample Assessment Material for first teaching September 2017
Time: 1 hour 30 minutes

Paper Reference
9FM0/3C
9FM0/4C

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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S 5 4 4 4 2 A 0 1 2 5

Pearson

Answer ALL questions. Write your answers in the spaces provided.

Unless otherwise indicated, whenever a numerical value of g is required, take $g = 9.8\,\mathrm{m\,s^{-2}}$ and give your answer to either 2 significant figures or 3 significant figures.

1. A particle P of mass 0.5 kg is moving with velocity $(4\mathbf{i} + \mathbf{j})\,\mathrm{m\,s^{-1}}$ when it receives an impulse $(2\mathbf{i} - \mathbf{j})\,\mathrm{N\,s}$.

Show that the kinetic energy gained by P as a result of the impulse is 12 J.

(6)

Question 1 continued

(Total for Question 1 is 6 marks)

2. A parcel of mass 5 kg is projected with speed $8\,\mathrm{m\,s^{-1}}$ up a line of greatest slope of a fixed rough inclined ramp.

The ramp is inclined at angle α to the horizontal, where $\sin\alpha = \frac{1}{7}$

The parcel is projected from the point A on the ramp and comes to instantaneous rest at the point B on the ramp, where $AB = 14$ m.

The coefficient of friction between the parcel and the ramp is μ.

In a model of the parcel's motion, the parcel is treated as a particle.

(a) Use the work-energy principle to find the value of μ.

(5)

(b) Suggest one way in which the model could be refined to make it more realistic.

(1)

Question 2 continued

(Total for Question 2 is 6 marks)

3. A particle of mass m kg lies on a smooth horizontal surface.

Initially the particle is at rest at a point O between two fixed parallel vertical walls.

The point O is equidistant from the two walls and the walls are 4 m apart.

At time $t = 0$ the particle is projected from O with speed u m s^{-1} in a direction perpendicular to the walls.

The coefficient of restitution between the particle and each wall is $\frac{3}{4}$

The magnitude of the impulse on the particle due to the first impact with a wall is λmu N s.

(a) Find the value of λ.

(3)

The particle returns to O, having bounced off each wall once, at time $t = 7$ seconds.

(b) Find the value of u.

(5)

Question 3 continued

(Total for Question 3 is 8 marks)

4.

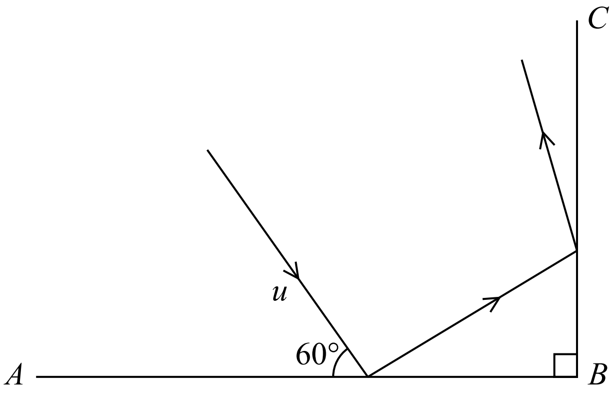


Figure 1

Figure 1 represents the plan view of part of a horizontal floor, where *AB* and *BC* are perpendicular vertical walls.

The floor and the walls are modelled as smooth.

A ball is projected along the floor towards *AB* with speed $u \, \text{m s}^{-1}$ on a path at an angle of 60° to *AB*. The ball hits *AB* and then hits *BC*.

The ball is modelled as a particle.

The coefficient of restitution between the ball and wall *AB* is $\frac{1}{\sqrt{3}}$

The coefficient of restitution between the ball and wall *BC* is $\sqrt{\frac{2}{5}}$

(a) Show that, using this model, the final kinetic energy of the ball is 35% of the initial kinetic energy of the ball.

(8)

(b) In reality the floor and the walls may not be smooth. What effect will the model have had on the calculation of the percentage of kinetic energy remaining?

(1)

Question 4 continued

Question 4 continued

Question 4 continued

(Total for Question 4 is 9 marks)

5. A car of mass 600 kg is moving along a straight horizontal road.

At the instant when the speed of the car is v m s^{-1}, the resistance to the motion of the car is modelled as a force of magnitude $(200 + 2v)$ N.

The engine of the car is working at a constant rate of 12 kW.

(a) Find the acceleration of the car at the instant when $v = 20$

(4)

Later on the car is moving up a straight road inclined at an angle θ to the horizontal,

where $\sin \theta = \frac{1}{14}$

At the instant when the speed of the car is v m s^{-1}, the resistance to the motion of the car from non-gravitational forces is modelled as a force of magnitude $(200 + 2v)$ N.

The engine is again working at a constant rate of 12 kW.

At the instant when the car has speed w m s^{-1}, the car is decelerating at 0.05 m s^{-2}.

(b) Find the value of w.

(5)

Question 5 continued

(Total for Question 5 is 9 marks)

6. [In this question **i** and **j** are perpendicular unit vectors in a horizontal plane.]

A smooth uniform sphere A has mass $2m$ kg and another smooth uniform sphere B, with the same radius as A, has mass $3m$ kg.

The spheres are moving on a smooth horizontal plane when they collide obliquely.

Immediately before the collision the velocity of A is $(3\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$ and the velocity of B is $(-5\mathbf{i} + 2\mathbf{j})\,\text{m s}^{-1}$.

At the instant of collision, the line joining the centres of the spheres is parallel to **i**.

The coefficient of restitution between the spheres is $\frac{1}{4}$

(a) Find the velocity of B immediately after the collision.

(7)

(b) Find, to the nearest degree, the size of the angle through which the direction of motion of B is deflected as a result of the collision.

(2)

Question 6 continued

Question 6 continued

Question 6 continued

(Total for Question 6 is 9 marks)

7. A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity $3mg$.

The other end of the string is attached to a fixed point O on a ceiling.

The particle hangs freely in equilibrium at a distance d vertically below O.

(a) Show that $d = \dfrac{4}{3}a$.

(3)

The point A is vertically below O such that $OA = 2a$.

The particle is held at rest at A, then released and first comes to instantaneous rest at the point B.

(b) Find, in terms of g, the acceleration of P immediately after it is released from rest.

(3)

(c) Find, in terms of g and a, the maximum speed attained by P as it moves from A to B.

(5)

(d) Find, in terms of a, the distance OB.

(3)

Question 7 continued

Question 7 continued

Question 7 continued

(Total for Question 7 is 14 marks)

8. A particle P of mass $2m$ and a particle Q of mass $5m$ are moving along the same straight line on a smooth horizontal plane.

They are moving in opposite directions towards each other and collide directly.

Immediately before the collision the speed of P is $2u$ and the speed of Q is u.

The direction of motion of Q is reversed by the collision.

The coefficient of restitution between P and Q is e.

(a) Find the range of possible values of e.

(8)

Given that $e = \frac{1}{3}$

(b) show that the kinetic energy lost in the collision is $\frac{40mu^2}{7}$.

(5)

(c) Without doing any further calculation, state how the amount of kinetic energy lost in the collision would change if $e > \frac{1}{3}$

(1)

Question 8 continued

Question 8 continued

Question 8 continued

(Total for Question 8 is 14 marks)

TOTAL FOR PAPER IS 75 MARKS

Paper 3C/4C: Further Mechanics 1 Mark Scheme

Question	Scheme	Marks	AOs
1	Use Impulse-momentum principle	M1	2.1
	$2\mathbf{i}-\mathbf{j}=0.5\mathbf{v}-0.5(4\mathbf{i}+\mathbf{j})$	A1	1.1b
	$\frac{1}{2}\mathbf{v}=4\mathbf{i}-\frac{1}{2}\mathbf{j}$, $\mathbf{v}=8\mathbf{i}-\mathbf{j}$ (m s^{-1})	A1	1.1b
	Use of KE $=\frac{1}{2}m\lvert\mathbf{v}\rvert^2-\frac{1}{2}m\lvert\mathbf{u}\rvert^2$	M1	2.1
	$=\frac{1}{2}\times 0.5\times\{(64+1)-(16+1)\}$	A1	1.1b
	$=\frac{1}{4}\times 48=12$ (J) *	A1*	1.1b
		(6)	
			(6 marks)

Notes:

M1: Difference of terms & dimensionally correct
A1: Correct unsimplified equation
A1: cao
M1: Must be a difference of two terms
Must be dimensionally correct
A1: Correct unsimplified equation
A1*: Complete justification of given answer

Question	Scheme	Marks	AOs
2(a)	$R = 5g\cos\alpha \left(= 5g \times \frac{4\sqrt{3}}{7} = 48.497... \right)$	M1	3.4
	Force due to friction = $\mu \times 5g\cos\alpha$	M1	3.4
	Work-Energy equation	M1	3.4
	$\frac{1}{2} \times 5 \times 64 = 5 \times 9.8 \times 14\sin\alpha + 14\mu R$	A1	1.1b
	$\mu = 0.0913$ or 0.091	A1	1.1b
		(5)	
(b)	Appropriate refinement	B1	3.5c
		(1)	
			(6 marks)

Notes:

(a)

M1: Condone sin/cos confusion

M1: Use of $\mu \times$ their R

M1: Must be using work-energy. Requires all terms
Condone sin/cos confusion, sign errors and their R

A1: Correct in θ and μR

A1: Accept 0.0913 **or** 0.091

(b)

B1: e.g.
- do not model the parcel as a particle and therefore take air resistance into account
- take into account the dimensions/uniformity of the parcel

Question	Scheme	Marks	AOs
3(a)	Use NEL to find the speed of particle after the first impact $= eu = \frac{3}{4}u\frac{\pi}{2}$	B1	3.4
	Impulse = $\lambda mu = mv - mu = \pm\left[\frac{3}{4}mu - (-mu)\right]$	M1	3.1b
	$\lambda = \frac{7}{4}$	A1	1.1b
		(3)	
(b)	Use NEL to find the speed of the particle after the second impact $= \frac{3}{4}\times\frac{3}{4}u = \frac{9}{16}u$	B1	3.4
	Use of $s = vt$ to find total time	M1	3.1b
	$7 = \frac{2}{u} + \frac{4}{\frac{3}{4}u} + \frac{2}{\frac{9}{16}u}\left(= \frac{2}{u} + \frac{16}{3u} + \frac{32}{9u}\right)$	A1	1.1b
	Solve for *u*: $63u = 18 + 48 + 32$	M1	1.1b
	$u = \frac{98}{63} = \frac{14}{9}\left(= 1.\dot{5}\right)$	A1	1.1b
		(5)	
			(8 marks)

Notes:

(a)
B1: Using Newton's experimental law as a model to find the speed after the first impact
M1: Must be a difference of two terms, taking account of the change in direction of motion
A1: cao

(b)
B1: Using NEL as a model to find the speed after the second impact
M1: Needs to be used for at least one stage of the journey
A1: Ur equivalent
M1: Solve their linear equation for *u*
A1: Accept 1.56 or better

Question	Scheme	Marks	AOs
4(a)	Complete strategy to find the kinetic energy after the second impact	M1	3.1b
	Parallel to *AB* after collision: $u\cos 60°$	M1	3.1b
	Perpendicular to *AB* after collision: $\frac{1}{\sqrt{3}}u\sin 60°$	M1	3.4
	Components of velocity after first impact: $\frac{u}{2}, \frac{u}{2}$	A1	1.1b
	Parallel to *BC* after collision: $\frac{u}{2}$ $\left(u\times\frac{1}{\sqrt{3}}\sin 60°\right)$	M1	3.1b
	Perpendicular to *BC* after collision: $\sqrt{\frac{2}{5}}\times\frac{u}{2}\left(=\frac{1}{\sqrt{10}}u\right)$ $\left(\sqrt{\frac{2}{5}}\times u\cos 60°\right)$	M1	3.4
	Components of velocity after second impact: $\frac{u}{2}, \frac{u}{\sqrt{10}}$	A1	1.1b
	Final KE $=\frac{1}{2}m\left(\frac{u^2}{4}+\frac{u^2}{10}\right)$ $\left(=\frac{mu^2}{2}\times\frac{7}{20}\right)$		
	Fraction of initial KE $=\dfrac{\frac{mu^2}{2}\times\frac{7}{20}}{\frac{mu^2}{2}}=\frac{7}{20}=35\%$ *	A1*	2.2a
		(8)	
(b)	The answer is too large - rough surface means resistance so final speed will be lower	B1	3.5a
		(1)	
			(9 marks)

Notes:

(a)

M1: Use of CLM parallel to the wall. Condone sin/cos confusion

M1: Use NEL as a model to find the speed perpendicular to the wall. Condone sin/cos confusion

A1: Both components correct with trig substituted (seen or implied)

M1: Use of CLM parallel to the wall. Condone sin/cos confusion

M1: Use NEL as a model to find the speed perpendicular to the wall. Condone sin/cos confusion

A1: Both components correct with trig substituted (seen or implied)

M1: Correct expression for total KE using their components after 2nd collision

A1*: Obtain **given answer** with sufficient working to justify it

(b)

B1: Clear explanation of how the modelling assumption has affected the outcome

Question	Scheme	Marks	AOs
5(a)	Use of $P = Fv$: $F = \dfrac{12000}{20}$	B1	3.3
	Equation of motion: $F - (200 + 2v) = 600a$	M1	3.4
	$600 - 240 = 600a$	A1ft	1.1b
	$360 = 600a$, $a = 0.6$ (m s^{-2})	A1	1.1b
		(4)	
(b)	Equation of motion:	M1	3.3
	$\dfrac{12000}{w} - \left(200 + 2w\right) - 600g\sin\theta = -600 \times 0.05$	A1 A1	1.1b 1.1b
	3 term quadratic and solve: $2w^2 + 590w - 12000 = 0$	M1	1.1b
	$w = \dfrac{-590 + \sqrt{590^2 + 96000}}{4} = 19.1\,(\text{m s}^{-1})$	A1	1.1b
		(5)	
			(9 marks)

Notes:

(a)

B1: 600 or equivalent

M1: Use the model to form the equation of motion
Must include all terms .Condone sign errors

A1ft: Correct for their *F*

A1: cao

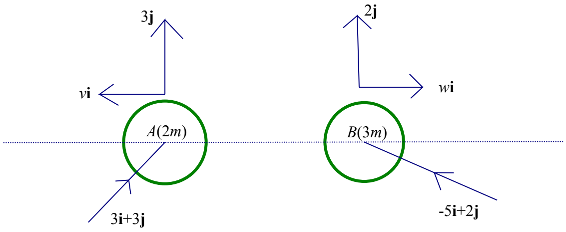
(b)

M1: Use the model to form the equation of motion
All terms needed. Condone sign errors and sin/cos confusion

A1: All correct A1A1
One error A1A0

M1: Dependent on the preceding M1. Use the equation of motion to form a 3-term quadratic in *w* only

A1: Accept 19. Do not accept more than 3 s.f.

Question	Scheme	Marks	AOs
6(a)			
	Overall strategy to find $\mathbf{v}_A$	M1	3.1a
	Velocity of A perpendicular to loc after collision $= 3\mathbf{j}$ (m s^{-1})	B1	3.4
	CLM parallel to loc	M1	3.1a
	$2m \times 3 - 3m \times 5 = 3mw - 2mv \quad (-9 = 3w\text{-}2v)$	A1	1.1b
	Correct use of impact law	M1	3.1a
	$v + w = \frac{1}{4}(3+5) \quad (=2)$	A1	1.1b
	Solve for w $\quad 3w - 2v = -9$ $2v + 2w = 4$		
	$\mathbf{v}_B = -\mathbf{i} + 2\mathbf{j}$ (m s^{-1}),	A1ft	1.1b
		(7)	
(b)	$\cos\theta = \frac{(-5\mathbf{i}+2\mathbf{j}).(-\mathbf{i}+2\mathbf{j})}{\sqrt{29}\sqrt{5}}$	M1	3.1a
	$\theta = 41.63...° = 42°$ (nearest degree)	A1	1.1b
	Alternative method: $\tan^{-1} 2 - \tan^{-1}\frac{2}{5} = 41.63...° = 42°$ (nearest degree)		
		(2)	
			(9 marks)

Notes:

(a)

M1: Correct overall strategy to form sufficient equations and solve for $\mathbf{v}_A$

B1: Use the model to find the component of $\mathbf{v}_A$ perpendicular to the line of centres

M1: Use CLM to form equation in v and w. Need all 4 terms, dimensionally correct

A1: Correct unsimplified

M1: Must be used the right way round

A1: Correct unsimplified

A1ft: $\mathbf{v}_B$ correct. Follow their $2\mathbf{j}$

(b)

M1: Complete method for finding the required angle. Follow their $\mathbf{v}_B$

A1: cao

Question	Scheme	Marks	AOs
7(a)	In equilibrium $\Rightarrow$ no resultant vertical force	M1	2.1
	$\frac{3mgx}{a} = mg$	A1	1.1b
	$x = \frac{a}{3}$, $d = \frac{4}{3}a$ *	A1*	2.2a
		(3)	
(b)	Equation of motion:	M1	3.1a
	$\frac{3mga}{a} - mg = m\ddot{x}$	A1	1.1b
	$\ddot{x} = 2g$	A1	1.1b
		(3)	
(c)	Max speed at equilibrium position	B1	3.1a
	Work energy & use of $\text{EPE} = \frac{\lambda x^2}{2a}$	M1	3.1a
	$\frac{3mga^2}{2a} = \frac{3mg\left(\frac{a}{3}\right)^2}{2a} + \frac{1}{2}mv^2 + mg\frac{2a}{3}$	A1 A1	1.1b 1.1b
	$\frac{1}{2}v^2 = ga\left(\frac{3}{2} - \frac{1}{6} - \frac{2}{3}\right) = \frac{2}{3}ga$, $v = \sqrt{\frac{4ga}{3}}$	A1	1.1b
		(5)	
(d)	At max ht. KE = 0. EPE lost = GPE gained	M1	3.1a
	$\frac{3mga^2}{2a} = mgh$	A1	1.1b
	$OB = \frac{a}{2}$	A1	1.1b
		(3)	
			(14 marks)

Question 7 notes:

(a)

M1: Use $T = \dfrac{\lambda x}{a}$ to form equation for equilibrium

A1: Correct unsimplified equation

A1*: Requires sufficient working to justify given answer plus a 'statement' that the required result has been achieved

(b)

M1: Use $T = \dfrac{\lambda x}{a}$ to form equation of motion
Need all 3 terms. Condone sign errors

A1: Correct unsimplified equation

A1: cao

(c)

B1: Seen or implied

M1: Form work-energy equation. All 4 terms needed
Condone sign errors

A1: Correct unsimplified equation A1A1
One error in the equation A1A0

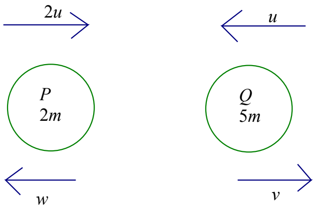
A1: cao

(d)

M1: Form energy equation

A1: Correct unsimplified equation

A1: cao

Question	Scheme	Marks	AOs
8(a)			
	Complete overall strategy to find v	M1	3.1a
	Use of CLM	M1	3.1a
	$2m\times 2u-5m\times u=5m\times v-2m\times w$, $(-u=5v-2w)$	A1	1.1b
	Use of Impact law:	M1	3.1a
	$v+w=e(2u+u)$	A1	1.1b
	Solve for v: $-u=5v-2w$ $6eu=2v+2w$		
	$7v=u(6e-1)$ $\left(v=\frac{u}{7}(6e-1)\right)$	A1	1.1b
	Direction of Q reversed: $v>0$	M1	3.4
	$\Rightarrow 1\ge e>\frac{1}{6}$	A1	1.1b
		(8)	
(b)	$e=\frac{1}{3}\quad\Rightarrow\quad v=\frac{u}{7}\ ,\quad w=\frac{6u}{7}$	B1	2.1
	Equation for KE lost	M1	2.1
	$\frac{1}{2}\times 2m\left(4u^2-\frac{36u^2}{49}\right)+\frac{1}{2}\times 5m\left(u^2-\frac{u^2}{49}\right)$	A1 A1	1.1b 1.1b
	$\frac{1}{2}mu^2\left(8-\frac{72}{49}+5-\frac{5}{49}\right)=\frac{40mu^2}{7}$ *	A1*	2.2a
		(5)	
(c)	Increase $e\Rightarrow$ more elastic $\Rightarrow$less energy lost	B1	2.2a
		(1)	
			(14 marks)

Question 8 notes:	
(a)	
M1:	Complete strategy to form sufficient equations in v and w and solve for v
M1:	Use CLM to form equation in v and w Needs all 4 terms & dimensionally correct
A1:	Correct unsimplified equation
M1:	Use NEL as a model to form a second equation in v and w. Must be used the right way round
A1:	Correct unsimplified equation
A1:	for v or $7v$ correct
M1:	Use the model to form a correct inequality for their v
A1:	Both limits required
(b)	
B1:	Or equivalent statements
M1:	Terms of correct structure combined correctly
A1:	Fully correct unsimplified A1A1 One error on unsimplified expression A1A0
A1*:	cso. plus a 'statement' that the required result has been achieved
(c)	
B1:	"less energy lost" or equivalent

Write your name here

Surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number ☐☐☐☐☐

Candidate Number ☐☐☐☐

Further Mathematics

Advanced
Further Mathematics Option 2
Paper 4: Further Mechanics 2

Sample Assessment Material for first teaching September 2017

Time: 1 hour 30 minutes

Paper Reference

9FM0/4F

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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S 5 4 4 4 6 A 0 1 2 3

Answer ALL questions. Write your answers in the spaces provided.

Unless otherwise indicated, whenever a numerical value of g is required, take $g = 9.8\,\text{m}\,\text{s}^{-2}$ and give your answer to either 2 significant figures or 3 significant figures.

1. A flag pole is 15 m long.

The flag pole is non-uniform so that, at a distance x metres from its base, the mass per unit length of the flag pole, $m\,\text{kg}\,\text{m}^{-1}$ is given by the formula $m = 10\left(1 - \frac{x}{25}\right)$.

The flag pole is modelled as a rod.

(a) Show that the mass of the flag pole is 105 kg.

(3)

(b) Find the distance of the centre of mass of the flag pole from its base.

(4)

Question 1 continued

(Total for Question 1 is 7 marks)

2.

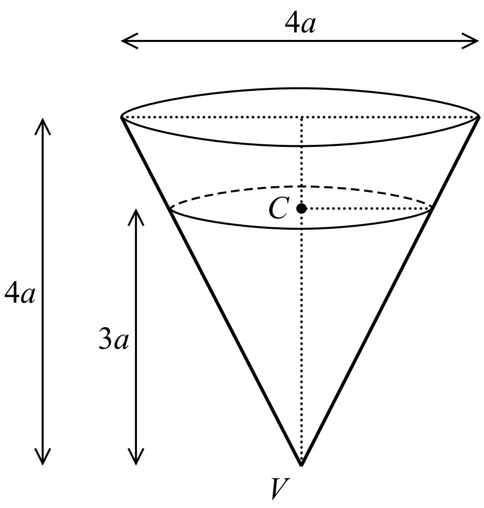

Figure 1

A hollow right circular cone, of base diameter $4a$ and height $4a$ is fixed with its axis vertical and vertex V downwards, as shown in Figure 1.

A particle of mass m moves in a horizontal circle with centre C on the rough inner surface of the cone with constant angular speed ω.

The height of C above V is $3a$.

The coefficient of friction between the particle and the inner surface of the cone is $\frac{1}{4}$.

Find, in terms of a and g, the greatest possible value of ω.

(8)

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Question 2 continued

(Total for Question 2 is 8 marks)

3.

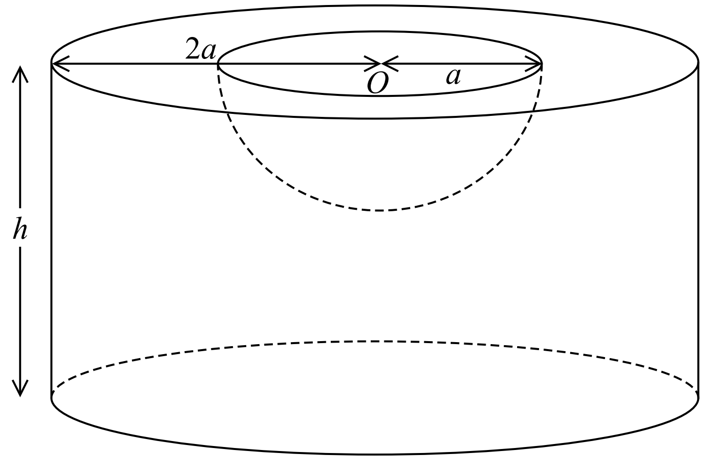

Figure 2

A uniform solid cylinder has radius $2a$ and height h ($h > a$).

A solid hemisphere of radius a is removed from the cylinder to form the vessel V.

The plane face of the hemisphere coincides with the upper plane face of the cylinder.

The centre O of the hemisphere is also the centre of the upper plane face of the cylinder, as shown in Figure 2.

(a) Show that the centre of mass of V is $\dfrac{3(8h^2 - a^2)}{8(6h - a)}$ from O.

(5)

The vessel V is placed on a rough plane which is inclined at an angle ϕ to the horizontal.

The lower plane circular face of V is in contact with the inclined plane.

Given that $h = 5a$, the plane is sufficiently rough to prevent V from slipping and V is on the point of toppling,

(b) find, to three significant figures, the size of the angle ϕ.

(4)

Question 3 continued

Question 3 continued

Question 3 continued

(Total for Question 3 is 9 marks)

4. A car of mass 500 kg moves along a straight horizontal road.

The engine of the car produces a constant driving force of 1800 N.

The car accelerates from rest from the fixed point O at time $t = 0$ and at time t seconds the car is x metres from O, moving with speed $v\,\mathrm{m\,s^{-1}}$.

When the speed of the car is $v\,\mathrm{m\,s^{-1}}$, the resistance to the motion of the car has magnitude $2v^2$ N.

At time T seconds, the car is at the point A, moving with speed $10\,\mathrm{m\,s^{-1}}$.

(a) Show that $T = \frac{25}{6}\ln 2$

(6)

(b) Show that the distance from O to A is $125\ln\frac{9}{8}$ m.

(5)

Question 4 continued

(Total for Question 4 is 11 marks)

5.

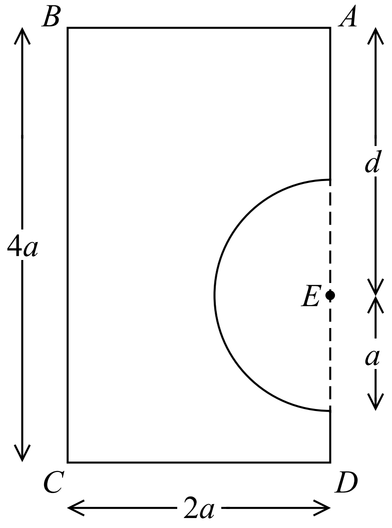

Figure 3

A shop sign is modelled as a uniform rectangular lamina *ABCD* with a semicircular lamina removed.

The semicircle has radius a, $BC = 4a$ and $CD = 2a$.

The centre of the semicircle is at the point E on AD such that $AE = d$, as shown in Figure 3.

(a) Show that the centre of mass of the sign is $\dfrac{44a}{3(16-\pi)}$ from AD.

(4)

The sign is suspended using vertical ropes attached to the sign at A and at B and hangs in equilibrium with AB horizontal.

The weight of the sign is W and the ropes are modelled as light inextensible strings.

(b) Find, in terms of W and π, the tension in the rope attached at B.

(2)

The rope attached at B breaks and the sign hangs freely in equilibrium suspended from A, with AD at an angle α to the downward vertical.

Given that $\tan \alpha = \dfrac{11}{18}$

(c) find d in terms of a and π.

(6)

Question 5 continued

Question 5 continued

Question 5 continued

(Total for Question 5 is 12 marks)

6. A small bead B of mass m is threaded on a circular hoop.

The hoop has centre O and radius a and is fixed in a vertical plane.

The bead is projected with speed $\sqrt{\frac{7}{2}ga}$ from the lowest point of the hoop.

The hoop is modelled as being smooth.

When the angle between OB and the downward vertical is θ, the speed of B is v.

(a) Show that $v^2 = ga\left(\frac{3}{2} + 2\cos\theta\right)$

(3)

(b) Find the size of θ at the instant when the contact force between B and the hoop is first zero.

(5)

(c) Give a reason why your answer to part (b) is not likely to be the actual value of θ.

(1)

(d) Find the magnitude and direction of the acceleration of B at the instant when B is first at instantaneous rest.

(5)

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DO NOT WRITE IN THIS AREA

Question 6 continued

Question 6 continued

Question 6 continued

(Total for Question 6 is 14 marks)

7. Two points A and B are 6 m apart on a smooth horizontal surface.

A light elastic string of natural length 2 m and modulus of elasticity 20 N, has one end attached to the point A.

A second light elastic string of natural length 2 m and modulus of elasticity 50 N, has one end attached to the point B.

A particle P of mass 3.5 kg is attached to the free end of each string.

The particle P is held at the point on AB which is 2 m from B and then released from rest.

In the subsequent motion both strings remain taut.

(a) Show that P moves with simple harmonic motion about its equilibrium position.
(7)

(b) Find the maximum speed of P.
(2)

(c) Find the length of time within each oscillation for which P is closer to A than to B.
(5)

Question 7 continued

Question 7 continued

Question 7 continued

(Total for Question 7 is 14 marks)

TOTAL FOR PAPER IS 75 MARKS

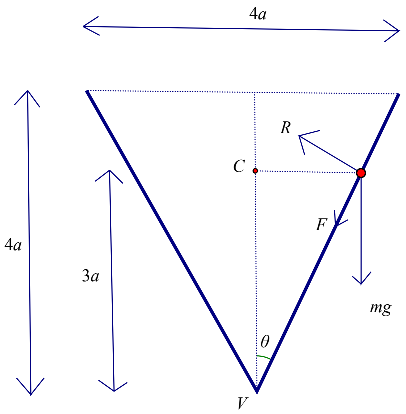
Paper 4F: Further Mechanics 2 Mark Scheme

Question	Scheme	Marks	AOs
1(a)	Total mass $= \int_0^{15} 10\left(1-\frac{x}{25}\right)\mathrm{d}x$	M1	2.1
	$=\left[10x-\frac{x^2}{5}\right]_0^{15}$	A1	1.1b
	$=150-\frac{225}{5}=105\,(\text{kg})$ *	A1*	1.1b
		(3)	
(b)	Taking moments about the base: $\int_0^{15} 10x\left(1-\frac{x}{25}\right)\mathrm{d}x$	M1	3.4
	$=\left[5x^2-\frac{2}{15}x^3\right]_0^{15}\ (=675)$	A1	1.1b
	$\Rightarrow 105d = 675$	M1	3.4
	$d = 6.43$ (m) $6\frac{3}{7}$ (m)	A1	1.1b
		(4)	
			(7 marks)

Notes:

(a)
M1: Use integration (usual rules)
A1: Correct integration
A1*: Use limits and show sufficient working to justify given answer

(b)
M1: Use the model to find the moment about the base (usual rules for integration)
A1: Correct integration
M1: Use the model to complete the moments equation
Require 105 and their 675 used correctly
A1: 6.43 or better

Question	Scheme	Marks	AOs
2			
	Complete overall strategy	M1	3.1b
	Resolve vertically	M1	3.3
	$mg + F\cos\theta = R\sin\theta$	A1	1.1b
	Horizontal equation of motion	M1	3.3
	$mr\omega^2 = R\cos\theta + F\sin\theta$	A1	1.1b
	Use of limiting friction since maximum ω	M1	3.3
	Substitute for trig ratios: $\frac{3a\omega^2}{2g} = \frac{9}{2}$	M1	1.1b
	Maximum $\omega = \sqrt{\frac{3g}{a}}$	A1	1.1b
			(8 marks)

Notes:

M1: Overall strategy to form equation in ω only e.g.
consider vertical and horizontal motrion and limiting friction

M1: Needs all 3 terms. Condone sign errors and sin/cos confusion

A1: Correct unsimplified equation

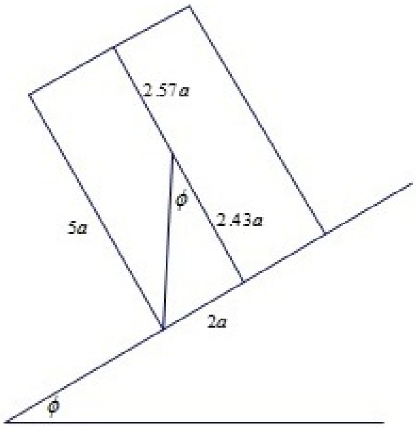
M1: Needs all 3 terms. Condone sign errors and sin/cos confusion

A1: Correct unsimplified equation

M1: Seen or implied

M1: Substitute to achieve equation in a, ω and g only

A1: Or equivalent exact form

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td rowspan="6">3(a)</td><td>
<table>
<tr><th></th><th>mass</th><th>c of m from O</th></tr>
<tr><td>cylinder</td><td>$4\pi a^2 h$</td><td>$\frac{h}{2}$</td></tr>
<tr><td>hemisphere</td><td>$\frac{2}{3}\pi a^3$</td><td>$\frac{3}{8}a$</td></tr>
<tr><td>V</td><td>$4\pi a^2 h-\frac{2}{3}\pi a^3$</td><td>$d$</td></tr>
</table>
</td><td></td><td></td></tr>
<tr><td>Mass ratios</td><td>B1</td><td>1.2</td></tr>
<tr><td>Correct distances</td><td>B1</td><td>1.2</td></tr>
<tr><td>Moments about a diameter through O</td><td>M1</td><td>2.1</td></tr>
<tr><td>$4\pi a^2 h\times\frac{h}{2}-\frac{2}{3}\pi a^3\times\frac{3}{8}a=2\pi a^2\left(2h-\frac{1}{3}a\right)\times d$</td><td>A1</td><td>1.1b</td></tr>
<tr><td>$d=\frac{h^2-\frac{a^2}{8}}{2h-\frac{a}{3}}=\frac{3\left(8h^2-a^2\right)}{8\left(6h-a\right)}$ *</td><td>A1*</td><td>2.2a</td></tr>
<tr><td></td><td>(5)</td><td></td></tr>
<tr><td rowspan="6">(b)</td><td>
</td><td></td><td></td></tr>
<tr><td>$h=5a\Rightarrow d=2.573...a$</td><td>B1</td><td>1.1b</td></tr>
<tr><td>About to topple so c of m above tipping point</td><td>M1</td><td>2.2a</td></tr>
<tr><td>$\Rightarrow\tan\phi=\frac{2a}{5a-2.573a}$</td><td>A1ft</td><td>1.1b</td></tr>
<tr><td>$\phi=39.5°$ or 0.689 rads</td><td>A1</td><td>1.1b</td></tr>
<tr><td></td><td>(4)</td><td></td></tr>
<tr><td colspan="4">(9 marks)</td></tr>
</table>

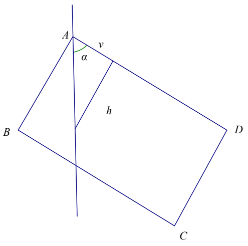
Question 3 notes:
(a) **B1:** Correct mass ratios **B1:** Correct distances **M1:** All three terms & dimensionally correct. Could use a parallel axis but final answer must be for the distance from O **A1:** Correct unsimplified equation **A1*:** Deduce the given answer. Their working must make it clear how they reached their answer
(b) **B1:** Distance of com from base **M1:** Condone tan the wrong way up **A1ft:** Correct unsimplified expression for trig ratio for ϕ following their d **A1:** 39.5° or 0.689 rads

Question	Scheme	Marks	AOs
4(a)	Equation of motion: $1800 - 2v^2 = 500a$ (when seen)	B1	2.1
	Select form for a: $= 500\dfrac{\mathrm{d}v}{\mathrm{d}t}$	M1	2.5
	$\int \dfrac{2}{500}\mathrm{d}t = \int \dfrac{1}{900 - v^2}\mathrm{d}v = \dfrac{1}{60}\int \dfrac{1}{30+v} + \dfrac{1}{30-v}\mathrm{d}v$	M1	2.1
	$\dfrac{t}{250} = \dfrac{1}{60}\ln(30+v) - \dfrac{1}{60}\ln(30-v)\ \ (+C)$	A1	1.1b
	$T = \dfrac{25}{6}\ln\left(\dfrac{30+10}{30-10}\right) = \dfrac{25}{6}\ln 2$ *	M1 A1*	2.1 2.2a
		(6)	
(b)	Equation of motion: $500v\dfrac{\mathrm{d}v}{\mathrm{d}x} = 1800 - 2v^2$	M1	2.5
	$\int \dfrac{500v}{1800 - 2v^2}\mathrm{d}v = \int 1\mathrm{d}x$	M1	2.1
	$-125\ln\left(1800 - 2v^2\right) = x\ \ (+C)$	A1	1.1b
	Use boundary conditions: $x = -125 \ln 1600 + 125 \ln 1800$	M1	2.1
	$x = 125\ln\dfrac{9}{8}$ (m) *	A1*	2.2a
		(5)	
		(11 marks)	

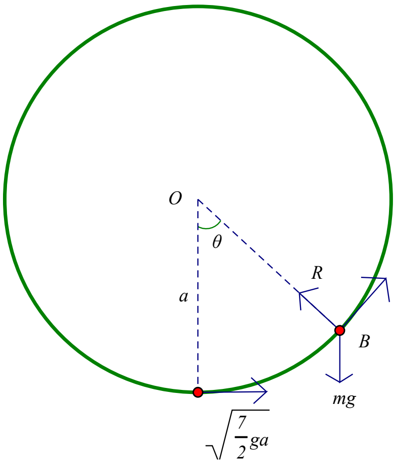
Notes:

(a)
B1: All three terms & dimensionally correct
M1: Use of correct form for acceleration to give equation in v, t only
M1: Separate variables and integrate
A1: Condone missing C
M1: Use boundary conditions correctly
A1*: Show sufficient working to justify given answer and a 'statement' that the required result has been achieved

(b)
M1: Correct form of acceleration in the equation of motion to give equation in v, x only
M1: Separate variables and integrate
A1: Condone missing C
M1: Extract and use boundary conditions
A1*: Show sufficient working to justify given answer and a 'statement' that the required result has been achieved

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td>**5(a)**</td><td>
<table>
<tr><th></th><th>Mass</th><th>From AD</th></tr>
<tr><td>Rectangle</td><td>$8a^2$</td><td>a</td></tr>
<tr><td>Semicircle</td><td>$\frac{1}{2}\pi a^2$</td><td>$\frac{4a}{3\pi}$</td></tr>
<tr><td>Sign</td><td>$a^2\left(8-\frac{\pi}{2}\right)$</td><td>$h$</td></tr>
</table>
</td><td></td><td></td></tr>
<tr><td></td><td>Mass ratios</td><td>B1</td><td>1.2</td></tr>
<tr><td></td><td>Moments about AD</td><td>M1</td><td>2.1</td></tr>
<tr><td></td><td>$a^2\left(8-\frac{\pi}{2}\right)h=8a^2\times a-\frac{1}{2}\pi a^2\times\frac{4a}{3\pi}\left(=8a^3-\frac{2}{3}a^3=\frac{22}{3}a^3\right)$</td><td>A1</td><td>1.1b</td></tr>
<tr><td></td><td>$\Rightarrow h=\frac{22}{3}a\div\left(8-\frac{\pi}{2}\right)=\frac{44a}{3(16-\pi)}$ *</td><td>A1*</td><td>2.2a</td></tr>
<tr><td></td><td></td><td>**(4)**</td><td></td></tr>
<tr><td>**(b)**</td><td>Moments about A
$2aT=\frac{44a}{3(16-\pi)}W$</td><td>M1</td><td>3.1b</td></tr>
<tr><td></td><td>$T=\frac{hW}{2a}=\frac{22W}{3(16-\pi)}$</td><td>A1</td><td>1.1b</td></tr>
<tr><td></td><td></td><td>**(2)**</td><td></td></tr>
<tr><td>**(c)**</td><td>
</td><td></td><td></td></tr>
<tr><td></td><td>Take moments about AB to find distance of com from AB</td><td>M1</td><td>3.1b</td></tr>
<tr><td></td><td>$8a^2\times 2a-\frac{1}{2}\pi a^2\times d=\left(8-\frac{1}{2}\pi\right)a^2\times v$</td><td>A1</td><td>1.1b</td></tr>
<tr><td></td><td>$v=\frac{32a-\pi d}{16-\pi}$</td><td>A1</td><td>1.1b</td></tr>
<tr><td></td><td>Correct trig for the given angle</td><td>M1</td><td>3.1b</td></tr>
<tr><td></td><td>$\tan\alpha=\frac{11}{18}=\frac{h}{v}=\frac{44a}{3(32a-\pi d)}$</td><td>A1ft</td><td>1.1b</td></tr>
<tr><td></td><td>$(24a=32a-\pi d,\quad 8a=\pi d)\qquad d=\frac{8a}{\pi}$</td><td>A1</td><td>1.1b</td></tr>
<tr><td></td><td></td><td>**(6)**</td><td></td></tr>
<tr><td colspan="4" align="right">**(12 marks)**</td></tr>
</table>

Question 5 notes:
(a) **B1:** Correct mass ratios **M1:** Need all three terms, must be dimensionally correct **A1:** Correct unsimplified equation **A1*:** Show sufficient working to justify the given answer and a 'statement' that the required result has been achieved
(b) **M1:** Could also take moments about B **or** about the c.o.m. and use **A1:** cso
(c) **M1:** All terms and dimensionally correct **A1:** Correct unsimplified equation **A1:** Or equivalent **M1:** Condone tan the wrong way up **A1:** Equation in a and d; follow through on their v **A1:** cao

Question	Scheme	Marks	AOs
6(a)			
	Conservation of energy	M1	2.1
	$\frac{1}{2}mv^2 + mga(1-\cos\theta) = \frac{1}{2}m\left(\frac{7}{2}ga\right)$	A1	1.1b
	$v^2 = ga\left(\frac{3}{2} + 2\cos\theta\right)$ *	A1*	2.2a
		(3)	
(b)	Resolve parallel to *OB* and use $\frac{mv^2}{a}$	M1	3.1b
	$R - mg\cos\theta = \frac{mv^2}{a}$	A1	1.1b
	Use R= 0 $g\cos\theta = -\frac{v^2}{a}$	M1	3.1b
	Solve for θ $\Rightarrow$ $g\cos\theta = -g\left(\frac{3}{2} + 2\cos\theta\right)$	M1	1.1b
	$\theta = 120°$	A1	1.1b
		(5)	
(c)	Any appropriate comment e.g. the hoop is unlikely to be smooth	B1	3.5b
		(1)	

Question	Scheme	Marks	AOs
6(d)	At rest $\Rightarrow \quad v=0$	M1	3.1b
	$\Rightarrow \quad \cos\theta = -\frac{3}{4}$	A1	1.1b
	Acceleration is tangential	M1	3.1b
	Magnitude $\lvert g\cos(\theta-90)\rvert = 6.48 \text{ m s}^{-2}$ or $\frac{\sqrt{7}}{4}g$	A1	1.1b
	At $\left(\cos^{-1}\left(-\frac{3}{4}\right)-90=\right)48.6°$ to the downward vertical	A1	1.1b
		(5)	
			(14 marks)

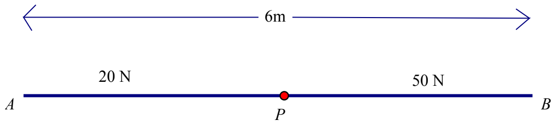
Question 6 notes:

(a)
M1: All terms required. Must be dimensionally correct
A1: Correct unsimplified equation
A1*: Show sufficient working to justify the given answer and a 'statement' that the required result has been achieved

(b)
M1: Resolve parallel to *OB*
A1: Correct equation
M1: Use $R=0$ seen or implied
M1: Solve for θ
A1: Accept $\frac{2\pi}{3}$

(c)
B1: Any appropriate comment e.g.
- hoop may not be smooth;
- air resistance could affect the motion

(d)
M1: $v = 0$ seen or implied
A1: Correct equation in θ
M1: Correct direction for acceleration
A1: Accept 6.48, 6.5 or exact in g
A1: Accept 0.848 (radians)

Question	Scheme	Marks	AOs
7(a)			
	$T_A = \frac{20e}{2}$, $T_B = \frac{50(2-e)}{2}$ e	M1	3.1a
	In equilibrium $T_A = T_B$, $10e = 25(2-e)$	M1	3.1a
	$(35e = 50)$, $e = \frac{10}{7}$	A1	1.1b
	Equation of motion for *P* when distance *x* from equilibrium position towards *B*:	M1	3.1a
	$3.5\ddot{x} = T_B - T_A = \frac{50(2-e-x)}{2} - \frac{20(e+x)}{2}$	A1 A1	1.1b 1.1b
	$= \frac{50\left(\frac{4}{7} - x\right)}{2} - \frac{20\left(\frac{10}{7} + x\right)}{2}$		
	$\Rightarrow 3.5\ddot{x} = -35x$, $\ddot{x} = -10x$ and hence SHM about the equilibrium position	A1	3.2a
		(7)	
(b)	Amplitude $= 2 - \frac{10}{7} = \frac{4}{7}$	B1 ft	2.2a
	Use of max speed $= a\omega$	M1	1.1b
	$= \frac{4}{7}\sqrt{10} = 1.81$ (m s^{-1})	A1 ft	1.1b
		(3)	

Question	Scheme	Marks	AOs
7(c)	Nearer to *A* than to *B*: $x < -\frac{3}{7}$	B1	3.1a
	Solve for $\sqrt{10}t$: $\cos\sqrt{10}t = -\frac{3}{4}$, $\sqrt{10}t = 2.418$............	M1	3.1a
	Length of time: $\frac{2}{\sqrt{10}}(\pi - 2.418...)$	M1	1.1b
	0.457 (seconds)	A1	1.1b
	Alternative: $\frac{3.864-2.419}{\sqrt{10}} = 0.457$		
	Alternative: $x = \frac{4}{7}\sin\sqrt{10}t = \frac{3}{7} \Rightarrow \sqrt{10}t = 0.8481$ or $\sqrt{10}t = 2.29353$ $t_1 = 0.2682,\ t_2 = 0.72527$ $\Rightarrow$ time = 0.457 (seconds)		
		(4)	
		(14 marks)	

Notes:

(a)

M1: Use of $T = \frac{\lambda x}{a}$

M1: Dependent on the preceding M1. Equate their tensions

A1: cao

M1: Condone sign error

A1: Correct unsimplified equation in *e* and *x* A1A1

Equation with one error A1A0

A1: Full working to justify conclusion that it is SHM about the equilibrium position

(b)

B1ft: Seen or implied. Follow their *e*

M1: Correct method for max. speed

A1ft: 1.81 or better. Follow their a, ω

(c)

B1: Seen or implied

M1: Use of $x = a\cos wt$

M1: Correct strategy for the required interval

A1: 0.457 or better

Pearson Edexcel Level 3 GCE

Further Mathematics

Advanced
Further Mathematics Option 1
Paper 3: Decision Mathematics 1
Further Mathematics Option 2
Paper 4: Decision Mathematics 1

Sample Assessment Material for first teaching September 2017

Time: 1 hour 30 minutes

Paper Reference
9FM0/3D
9FM0/4D

You must have:
Decision Mathematics Answer Book (enclosed), calculator

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Write your answers for this paper in the Decision Mathematics answer book provided.
- **Fill in the boxes** at the top of the answer book with your name, centre number and candidate number.
- Do not return the question paper with the answer book.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

S54443A

1/1/1/1/1/1/

Answer ALL questions. Write your answers in the answer book provided.

1. A list of n numbers needs to be sorted into descending order starting at the left-hand end of the list.

(a) Describe how to carry out the first pass of a bubble sort on the numbers in the list.

(2)

Bubble sort is a quadratic order algorithm.

A computer takes approximately 0.021 seconds to apply a bubble sort to a list of 2000 numbers.

(b) Estimate the time it would take the computer to apply a bubble sort to a list of 50 000 numbers. Make your method clear.

(2)

(Total for Question 1 is 4 marks)

2.

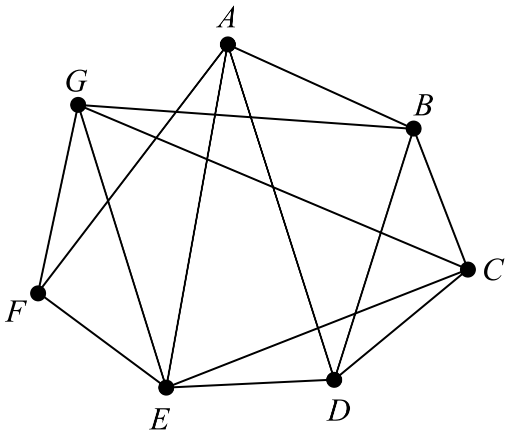


Figure 1

(a) Define what is meant by a **planar** graph.

(2)

(b) Starting at A, find a Hamiltonian cycle for the graph in Figure 1.

(1)

Arc AG is added to Figure 1 to create the graph shown in Figure 2.

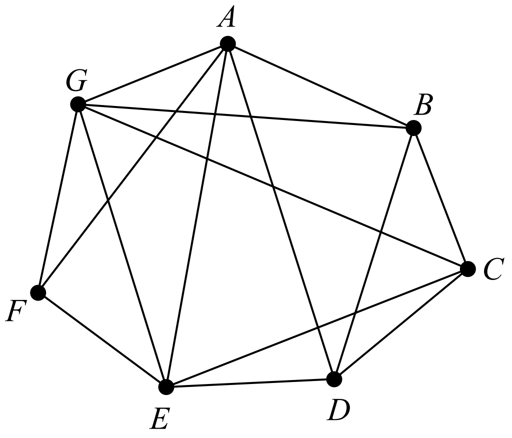


Figure 2

Taking ABCDEFGA as the Hamiltonian cycle,

(c) use the planarity algorithm to determine whether the graph shown in Figure 2 is planar. You must make your working clear and justify your answer.

(4)

(Total for Question 2 is 7 marks)

3. (a) Explain clearly the difference between the classical travelling salesperson problem and the practical travelling salesperson problem.

(2)

	A	B	C	D	E	F	G
A	–	17	24	16	21	18	41
B	17	–	35	25	30	31	x
C	24	35	–	28	20	35	32
D	16	25	28	–	29	19	45
E	21	30	20	29	–	22	35
F	18	31	35	19	22	–	37
G	41	x	32	45	35	37	–

The table shows the least distances, in km, by road between seven towns, A, B, C, D, E, F and G. The least distance between B and G is x km, where $x > 25$

Preety needs to visit each town at least once, starting and finishing at A. She wishes to minimise the total distance she travels.

(b) Starting by deleting B and all of its arcs, find a lower bound for Preety's route.

(3)

Preety found the nearest neighbour routes from each of A and C. Given that the sum of the lengths of these routes is 331 km,

(c) find x, making your method clear.

(4)

(d) Write down the smallest interval that you can be confident contains the optimal length of Preety's route. Give your answer as an inequality.

(2)

(Total for Question 3 is 11 marks)

4.

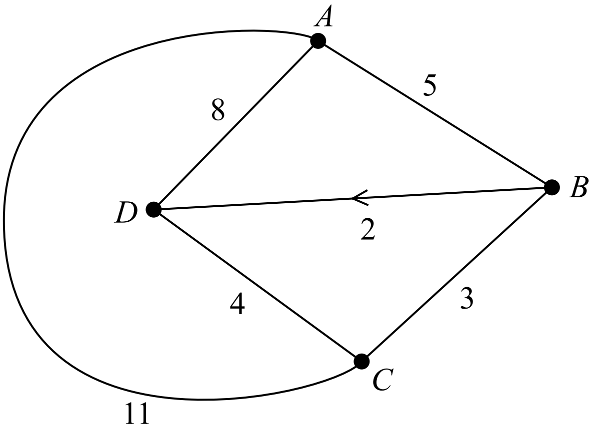


Figure 3

The network in Figure 3 shows the roads linking a depot, D, and three collection points A, B and C. The number on each arc represents the length, in miles, of the corresponding road. The road from B to D is a one-way road, as indicated by the arrow.

(a) Explain clearly if Dijkstra's algorithm can be used to find a route from D to A.

(1)

The initial distance and route tables for the network are given in the answer book.

(b) Use Floyd's algorithm to find a table of least distances. You should show both the distance table and the route table after each iteration.

(7)

(c) Explain how the final route table can be used to find the shortest route from D to B. State this route.

(2)

There are items to collect at A, B and C. A van will leave D to make these collections in any order and then return to D. A minimum route is required.

Using the final distance table and the Nearest Neighbour algorithm starting at D,

(d) find a minimum route and state its length.

(2)

Floyd's algorithm and Dijkstra's algorithm are applied to a network. Each will find the shortest distance between vertices of the network.

(e) Describe how the results of these algorithms differ.

(2)

(Total for Question 4 is 14 marks)

5. A garden centre makes hanging baskets to sell to its customers. Three types of hanging basket are made, *Sunshine*, *Drama* and *Peaceful*. The plants used are categorised as *Impact*, *Flowering* or *Trailing*.

Each *Sunshine* basket contains 2 *Impact* plants, 4 *Flowering* plants and 3 *Trailing* plants.
Each *Drama* basket contains 3 *Impact* plants, 2 *Flowering* plants and 4 *Trailing* plants.
Each *Peaceful* basket contains 1 *Impact* plant, 3 *Flowering* plants and 2 *Trailing* plants.

The garden centre can use at most 80 *Impact* plants, at most 140 *Flowering* plants and at most 96 *Trailing* plants each day.

The profit on *Sunshine*, *Drama* and *Peaceful* baskets are £12, £20 and £16 respectively. The garden centre wishes to maximise its profit.

Let x, y and z be the number of *Sunshine*, *Drama* and *Peaceful* baskets respectively, produced each day.

(a) Formulate this situation as a linear programming problem, giving your constraints as inequalities.

(5)

(b) State the further restriction that applies to the values of x, y and z in this context.

(1)

The Simplex algorithm is used to solve this problem. After one iteration, the tableau is

b.v.	x	y	z	r	s	t	Value
r	$-\frac{1}{4}$	0	$-\frac{1}{2}$	1	0	$-\frac{3}{4}$	8
s	$\frac{5}{2}$	0	2	0	1	$-\frac{1}{2}$	92
y	$\frac{3}{4}$	1	$\frac{1}{2}$	0	0	$\frac{1}{4}$	24
P	3	0	–6	0	0	5	480

(c) State the variable that was increased in the first iteration. Justify your answer.

(2)

(d) Determine how many plants in total are being used after only one iteration of the Simplex algorithm.

(1)

(e) Explain why for a second iteration of the Simplex algorithm the 2 in the z column is the pivot value.

(2)

After a second iteration, the tableau is

b.v.	x	y	z	r	s	t	Value
r	$\frac{3}{8}$	0	0	1	$\frac{1}{4}$	$-\frac{7}{8}$	31
s	$\frac{5}{4}$	0	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	46
y	$\frac{1}{8}$	1	0	0	$-\frac{1}{4}$	$\frac{3}{8}$	1
P	$\frac{21}{2}$	0	0	0	3	$\frac{7}{2}$	756

(f) Use algebra to explain why this tableau is optimal.

(1)

(g) State the optimal number of each type of basket that should be made.

(1)

The manager of the garden centre is able to increase the number of *Impact* plants available each day from 80 to 100. She wants to know if this would increase her profit.

(h) Use your final tableau to determine the effect of this increase. (You should not carry out any further calculations.)

(2)

(Total for Question 5 is 15 marks)

6.

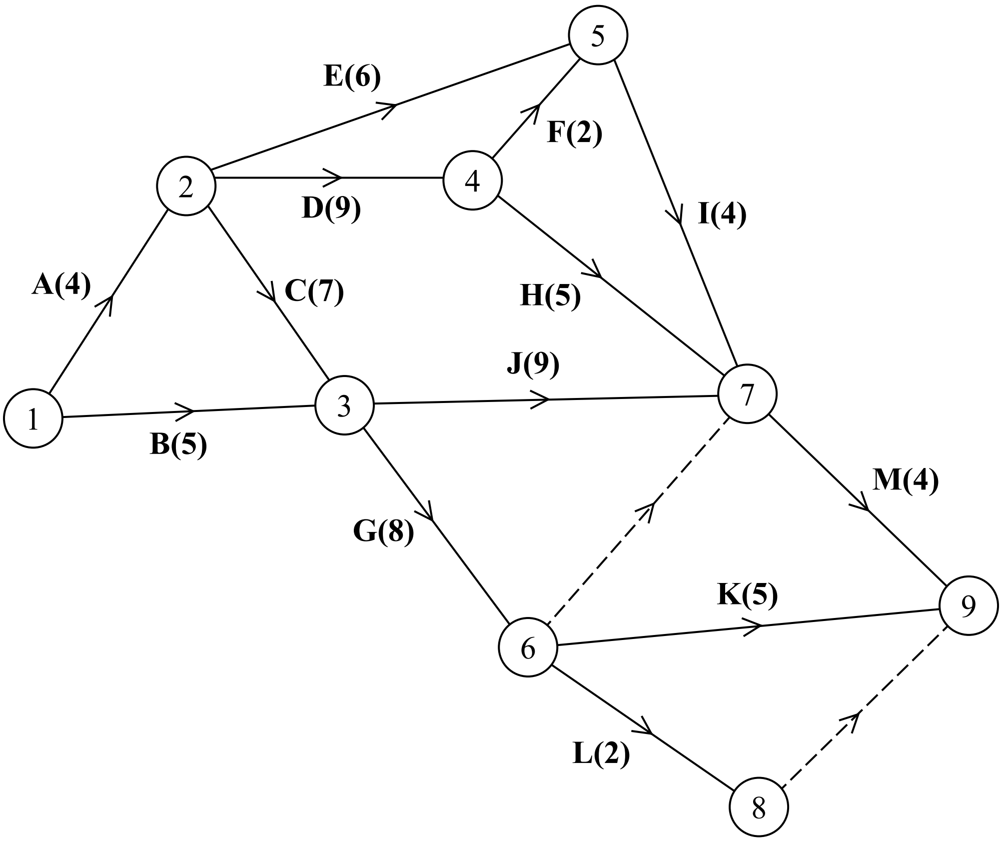


Figure 4

A project is modelled by the activity network shown in Figure 4. The activities are represented by the arcs. The number in brackets on each arc gives the time, in days, to complete that activity. Each activity requires one worker. The project is to be completed in the shortest possible time.

(a) Calculate the early time and the late time for each event, using Diagram 1 in the answer book.

(3)

(b) On Grid 1 in the answer book, complete the cascade (Gantt) chart for this project.

(3)

(c) On Grid 2 in the answer book, draw a resource histogram to show the number of workers required each day when each activity begins at its earliest time.

(3)

The supervisor of the project states that only three workers are required to complete the project in the minimum time.

(d) Use Grid 2 to determine if the project can be completed in the minimum time by only three workers. Give reasons for your answer.

(3)

(Total for Question 6 is 12 marks)

7. A linear programming problem in x, y and z is described as follows.

Maximise $P = 3x + 2y + 2z$

subject to $2x + 2y + z \leqslant 25$

$x + 4y \leqslant 15$

$x \geqslant 3$

(a) Explain why the Simplex algorithm cannot be used to solve this linear programming problem.

(1)

The big-M method is to be used to solve this linear programming problem.

(b) Define, in this context, what M represents. You must use correct mathematical language in your answer.

(1)

The initial tableau for a big-M solution to the problem is shown below.

b.v.	x	y	z	s_1	s_2	s_3	t_1	Value
s_1	2	2	1	1	0	0	0	25
s_2	1	4	0	0	1	0	0	15
t_1	1	0	0	0	0	–1	1	3
P	$-(3 + M)$	–2	–2	0	0	M	0	$-3M$

(c) Explain clearly how the equation represented in the b.v. t_1 row was derived.

(1)

(d) Show how the equation represented in the b.v. P row was derived.

(2)

The tableau obtained from the first iteration of the big-M method is shown below.

b.v.	x	y	z	s_1	s_2	s_3	t_1	Value
s_1	0	2	1	1	0	2	–2	19
s_2	0	4	0	0	1	1	–1	12
x	1	0	0	0	0	–1	1	3
P	0	–2	–2	0	0	–3	$3 + M$	9

(e) Solve the linear programming problem, starting from this second tableau. You must

- give a detailed explanation of your method by clearly stating the row operations you use and
- state the solution by deducing the final values of P, x, y and z.

(7)

(Total for Question 7 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS

Write your name here

Surname	Other names

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Level 3 GCE

Centre Number ☐☐☐☐☐

Candidate Number ☐☐☐☐

Further Mathematics

Advanced
Further Mathematics Option 1
Paper 3: Decision Mathematics 1
Further Mathematics Option 2
Paper 4: Decision Mathematics 1

Sample Assessment Material for first teaching September 2017

Paper Reference
9FM0/3D
9FM0/4D

Answer Book
Do not return the question paper with the answer book.

Total Marks

Turn over ▶

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1/1/1/1/1/1/

S 5 4 4 4 3 A 0 1 1 6

Pearson

1.

DO NOT WRITE IN THIS AREA

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Question 1 continued

(Total for Question 1 is 4 marks)

2.

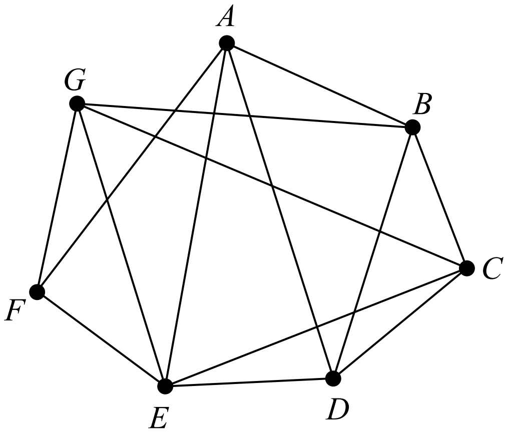


Figure 1

Question 2 continued

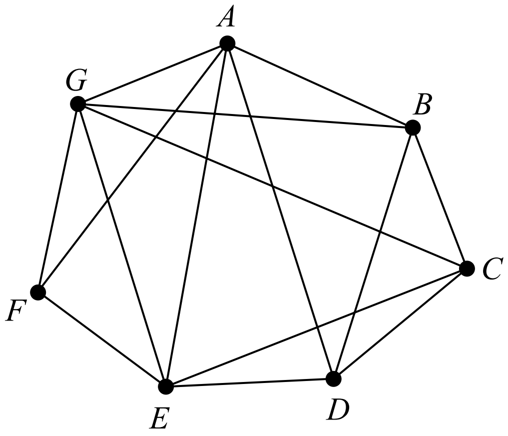


Figure 2

(Total for Question 2 is 7 marks)

3.

	A	B	C	D	E	F	G
A	–	17	24	16	21	18	41
B	17	–	35	25	30	31	x
C	24	35	–	28	20	35	32
D	16	25	28	–	29	19	45
E	21	30	20	29	–	22	35
F	18	31	35	19	22	–	37
G	41	x	32	45	35	37	–

Question 3 continued

	A	B	C	D	E	F	G
A	–	17	24	16	21	18	41
B	17	–	35	25	30	31	x
C	24	35	–	28	20	35	32
D	16	25	28	–	29	19	45
E	21	30	20	29	–	22	35
F	18	31	35	19	22	–	37
G	41	x	32	45	35	37	–

(Total for Question 3 is 11 marks)

4.

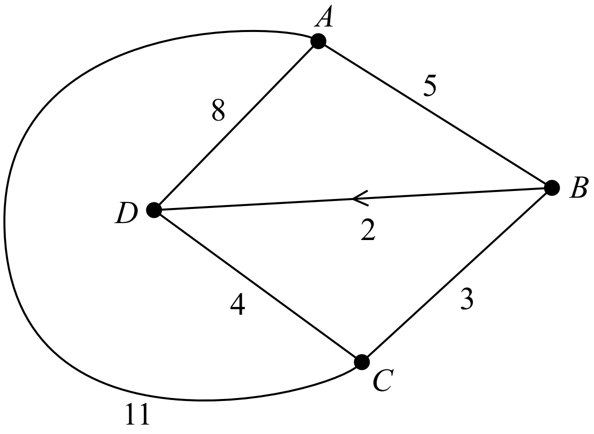


Figure 3

(b)

	A	**B**	**C**	**D**
A	–	5	11	8
B	5	–	3	2
C	11	3	–	4
D	8	∞	4	–

	A	**B**	**C**	**D**
A	A	B	C	D
B	A	B	C	D
C	A	B	C	D
D	A	B	C	D

	A	**B**	**C**	**D**
A				
B				
C				
D				

	A	**B**	**C**	**D**
A				
B				
C				
D				

	A	**B**	**C**	**D**
A				
B				
C				
D				

	A	**B**	**C**	**D**
A				
B				
C				
D				

Question 4 continued

	A	B	C	D
A				
B				
C				
D				

	A	B	C	D
A				
B				
C				
D				

	A	B	C	D
A				
B				
C				
D				

	A	B	C	D
A				
B				
C				
D				

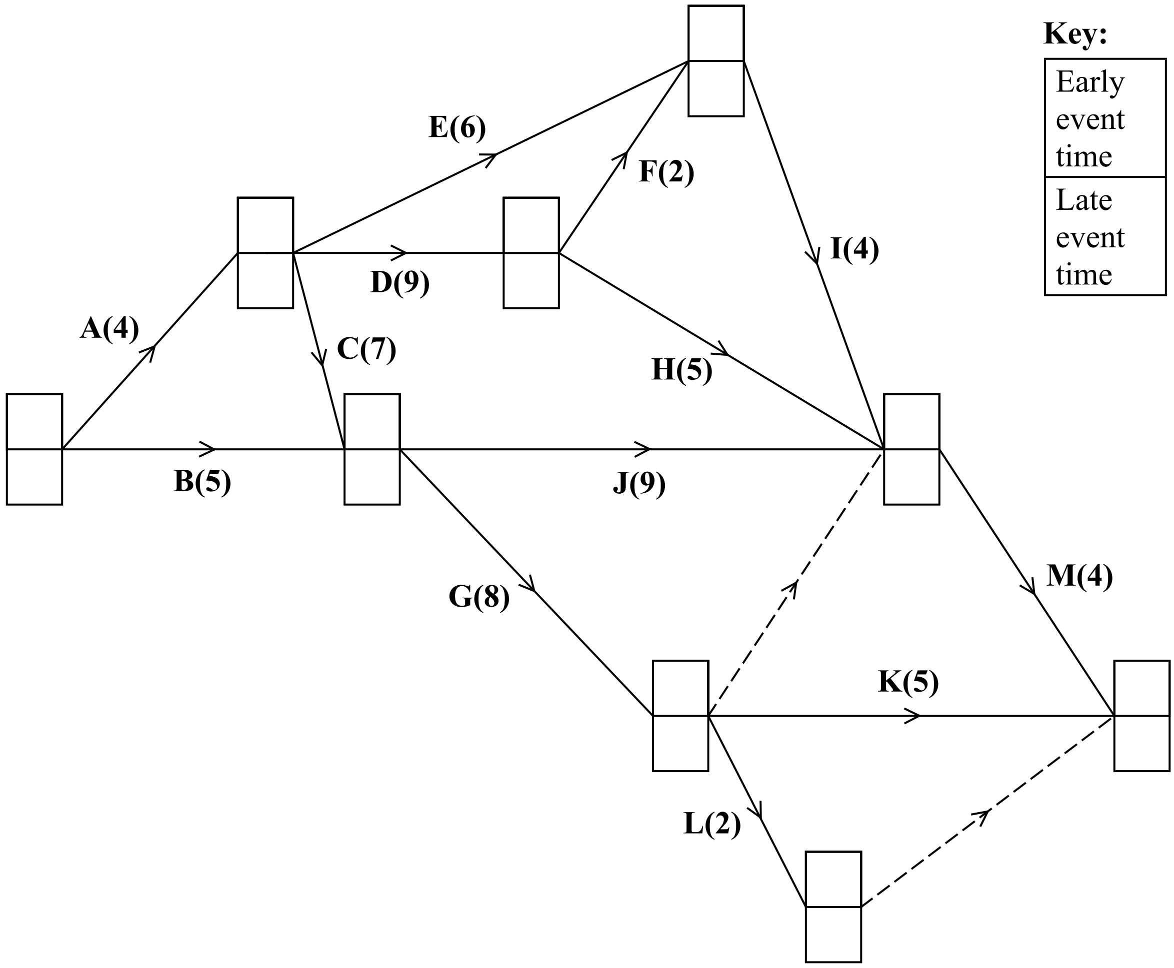
(Total for Question 4 is 14 marks)

5.

Question 5 continued

(Total for Question 5 is 15 marks)

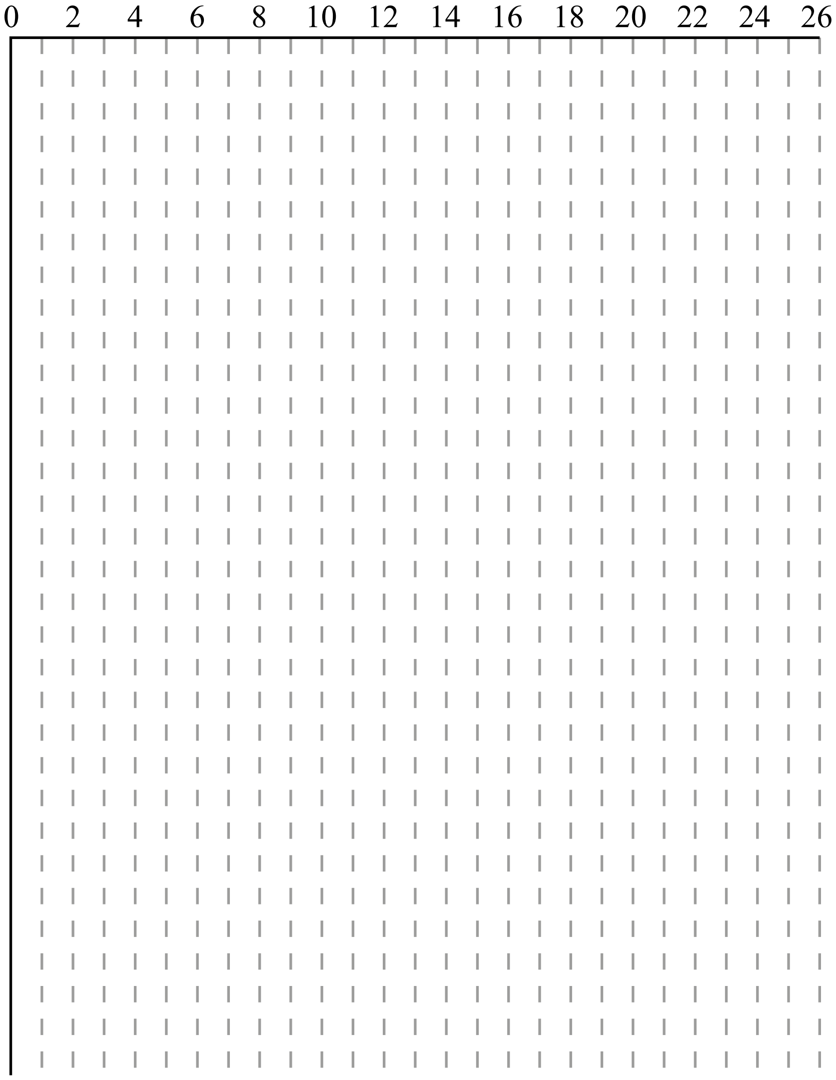
6.



Key:
Early event time
Late event time
A(4)
B(5)
C(7)
D(9)
E(6)
F(2)
G(8)
H(5)
I(4)
J(9)
K(5)
L(2)
M(4)

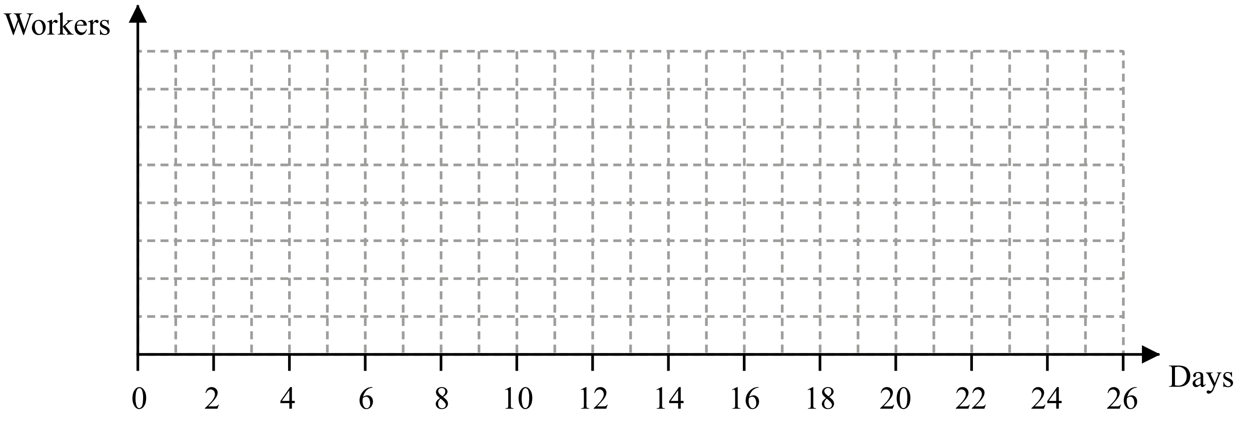
Diagram 1

Question 6 continued



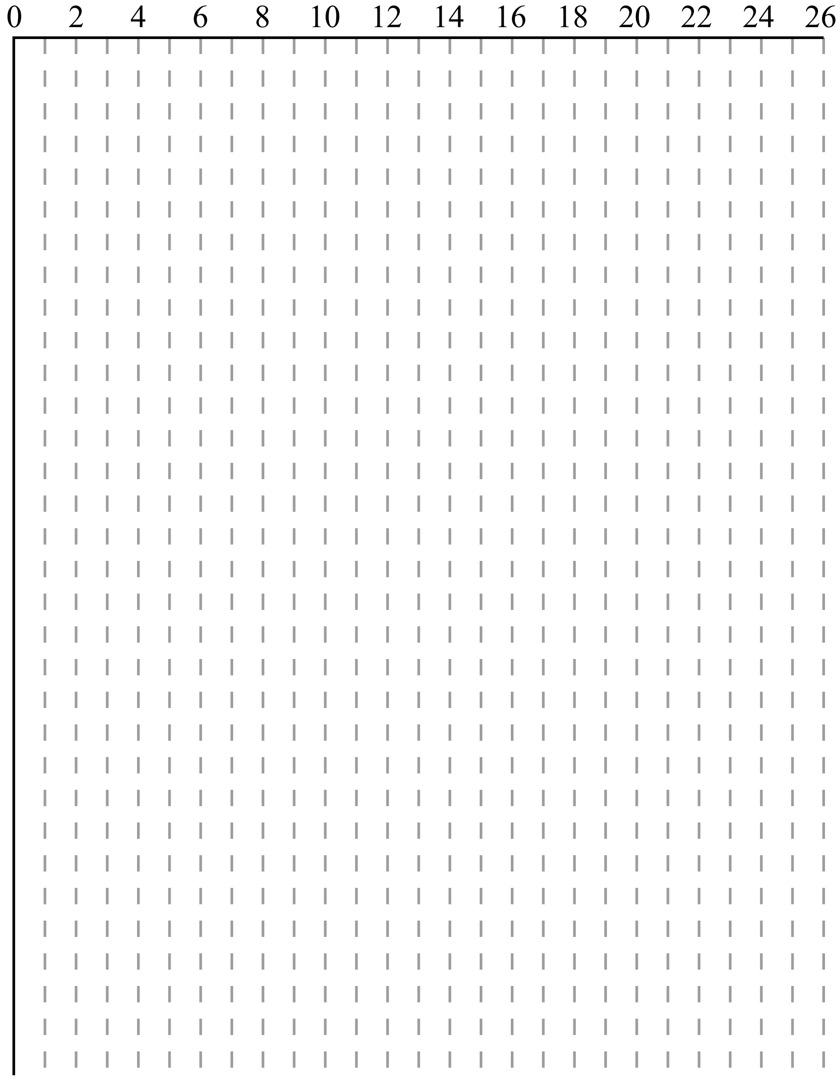
Grid 1

(There is a spare grid on the next page)



Grid 2

Question 6 continued



Copy of Grid 1

(Total for Question 6 is 12 marks)

7.

b.v.	x	y	z	s_1	s_2	s_3	t_1	Value
s_1	0	2	1	1	0	2	−2	19
s_2	0	4	0	0	1	1	−1	12
x	1	0	0	0	0	−1	1	3
P	0	−2	−2	0	0	−3	$3 + M$	9

b.v.	x	y	z	s_1	s_2	s_3	t_1	Value	Row Ops
P									

b.v.	x	y	z	s_1	s_2	s_3	t_1	Value	Row Ops
P									

Question 7 continued

b.v.	x	y	z	s_1	s_2	s_3	t_1	Value	Row Ops
P									

b.v.	x	y	z	s_1	s_2	s_3	t_1	Value	Row Ops
P									

(Total for Question 7 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS

Paper 3D/4D: Decision Mathematics 1 Mark Scheme

Question	Scheme	Marks	AOs
1(a)	In the first pass we compare the first value with the second value and we swap these values if the second is larger than the first	B1	2.4
	We then compare the value which is now second with the third value and swap if the third is larger than the second. We continue in this way until we reach the end of this list	B1	2.4
		(2)	
(b)	$t = 0.021 \times \left(\frac{50000}{2000}\right)^2$	M1	1.1a
	t = 13.125(seconds)	A1	1.1b
		(2)	
			(4 marks)

Notes:

(a)

B1: Comparing first value with second value, swap if second is larger (oe) – in their reasoning it must be clear that the first value in the list is being compared with the second value in the list and swapping if the second is larger than the first

B1: Compare second with third, (third with fourth), and so on until the end of the list – must be clear in their reasoning that after the first comparison the second value in the list is compared with the third value and so on until the end of the list

(b)

M1: Correct method seen – accept 25 for 50000/2000

A1: cao

Question	Scheme	Marks	AOs
2(a)	A planar graph is a graph that can be drawn so that…	B1	1.2
	… no arc meets another arc except at a vertex	B1	1.2
		(2)	
(b)	e.g. ABCDEGFA	B1	1.1b
		(1)	
(c)	Creates two lists of arcs	M1	2.1
	e.g. BG AD		
	CG BD	M1	1.1b
	EG AE		
	CE AF	A1	1.1b
	Since no arc appears in both lists, the graph is planar (**or** draws a planar version)	A1	2.4
		(4)	
			(7 marks)

Notes:

(a)

B1: A clear indication that a planar graph ‘can be drawn’ – allow this mark even if candidate implies that arcs can cross each other

B1: cao – no arc meets another arc except at a vertex – technical language must be correct

(b)

B1: Any correct Hamiltonian cycle (must start and finish at A) – must contain 8 vertices with every vertex appearing only once (except A)

(c)

M1: Creates two list of arcs (with at least three arcs in each list) which contain no common arcs

M1: Four arcs (in each list) and within each list there are no crossing arcs

A1: cao

A1: Correct reasoning that no arc appears in both lists + so the graph is therefore planar

Question	Scheme	Marks	AOs
3(a)	e.g. in the practical problem each vertex must be visited at least once. In the classical problem each vertex must be visited just once	B2,1,0	2.4 2.4
		(2)	
(b)	Prim's algorithm on reduced network starting at A: AD, AF, AE, CE, CG	M1	1.1b
	Lower bound = 107 + 17 + 25 = 149 (km)	M1 A1	1.1b 1.1b
		(3)	
(c)	NNA from A: A – D – F – E – C – G – B – A = $126+x$ NNA from C: C – E – A – D – F – B – G – C =$139+x$	M1 A1 A1	1.1b 1.1b 1.1b
	$(126+x)+(139+x)=331 \Rightarrow x=33$	A1	1.1b
		(4)	
(d)	$149 < \text{optimal} \leqslant 159$	M1 A1	2.2b 1.1b
		(2)	
			(11 marks)

Notes:

(a)

B1: Understands the difference is connected to the number of times each vertex may be visited (but maybe incorrectly attributed). Must be an attempt at a difference (so must refer to both the classical and practical problems explicitly). Technical language (vertex/node) must be correct. Need not imply each/every/all (oe) vertices for this first mark

B1: Correctly reasons which is classical and which is practical and correctly states the difference. Must imply that each/every/all (oe) vertices are visited, so for example, 'the practical problem visits a vertex at least once while the classical visits a vertex only once' is B1B0 (note that B0B1 is not possible in (a))

(b)

M1: Correctly applying Prim's algorithm from node A for the first four arcs (or five nodes)

M1: Candidates weight of their RMST + 17 + 25 (the two smallest arcs incident to B)

A1: cao (condone lack of units)

(c)

M1: Either one route, must return to A

A1: One correct route, must return to A and corresponding length correct (do not is in part (c) if correct lengths seen but are then doubled)

A1: Both routes correct and their corresponding lengths correct

A1: cao for x

(d)

M1: Their numbers correctly used, accept any inequalities or any indication of interval from their 149 to their 159 (so 149 – 159 can score this mark). This mark is dependent on two routes seen in (c), however, neither of the two totals need to be correct. Please note that UB > LB for this mark

A1: cao (no follow through on their values) including correct inequalities or equivalent set notation (but condone $149 \leqslant \text{optimal} \leqslant 159$)

Question	Scheme	Marks	AOs
4(a)	Yes Dijkstra's algorithm can be applied to either a directed or undirected network	B1	3.5b
		(1)	
(b)	Initial tables $\begin{bmatrix} - & 5 & 11 & 8 \\ 5 & - & 3 & 2 \\ 11 & 3 & - & 4 \\ 8 & \infty & 4 & - \end{bmatrix} \begin{bmatrix} A & B & C & D \\ A & B & C & D \\ A & B & C & D \\ A & B & C & D \end{bmatrix}$		
	1st iteration $\begin{bmatrix} - & 5 & 11 & 8 \\ 5 & - & 3 & 2 \\ 11 & 3 & - & 4 \\ 8 & [13] & 4 & - \end{bmatrix} \begin{bmatrix} A & B & C & D \\ A & B & C & D \\ A & B & C & D \\ A & [A] & C & D \end{bmatrix}$	M1 A1	1.1b 1.1b
	2nd iteration $\begin{bmatrix} - & 5 & [8] & [7] \\ 5 & - & 3 & 2 \\ [8] & 3 & - & 4 \\ 8 & 13 & 4 & - \end{bmatrix} \begin{bmatrix} A & B & [B] & [B] \\ A & B & C & D \\ [B] & B & C & D \\ A & A & C & D \end{bmatrix}$	M1 A1ft	1.1b 1.1b
	3rd iteration $\begin{bmatrix} - & 5 & 8 & 7 \\ 5 & - & 3 & 2 \\ 8 & 3 & - & 4 \\ 8 & [7] & 4 & - \end{bmatrix} \begin{bmatrix} A & B & B & B \\ A & B & C & D \\ B & B & C & D \\ A & [C] & C & D \end{bmatrix}$	M1 A1ft	1.1b 1.1b
	4th iteration $\begin{bmatrix} - & 5 & 8 & 7 \\ 5 & - & 3 & 2 \\ 8 & 3 & - & 4 \\ 8 & 7 & 4 & - \end{bmatrix} \begin{bmatrix} A & B & B & B \\ A & B & C & D \\ B & B & C & D \\ A & C & C & D \end{bmatrix}$ no changes therefore optimal	A1	1.1b
		(7)	
(c)	Start at D (4th) row and read across to the B (2nd) column, there is a C there		
	so the route starts DC. Look at the C row, B column and you see B	B1	2.4
	The route is therefore DCB	B1	2.2a
		(2)	

Question	Scheme	Marks	AOs
(d)	D – C –B – A **– B** – D	M1	2.2a
	Length 19 (miles)	A1	1.1b
		(2)	
(e)	Dijkstra's algorithm finds the shortest distances from **one** vertex to **all** the others. Floyd's algorithm finds the shortest distance between **every pair** of vertices.	B1 B1	2.5 2.5
		(2)	
			(14 marks)

Question 4 notes:

(a)
B1: cao (must include mention of 'directed' network)
M1: No change in the first row and first column of both tables with at least one value in the distance table reduced and one value in the route table changed

(b)
A1: cao
M1: No change in the second row and second column of both tables with at least two values in the distance table reduced and two values in the route table changed
A1ft: Correct second iteration follow through from the candidate's first iteration
M1: No change in the third row and third column of both tables with at least one value in the distance table reduced and one value in the route table changed
A1ft: Correct third iteration follow through from the candidate's second iteration
A1: cao (no change after the fourth iteration) – all previous marks must have been awarded in this part

(c)
B1: Clear indication of how the final route table can be used to get from D to B (therefore must mention the correct rows and columns in their reasoning)
B1: Completely correct argument + correct route (DCB)

(d)
M1: Deduce correctly their minimum route from their final distance table (dependent on all M marks in (a)) must begin and end at D
A1: cao (length of 19)

(e)
B1: cao – must use correct language '**one** vertex to **all** other vertices'
B1: cao – must use correct language '**every pair** of vertices'

Question	Scheme	Marks	AOs
5(a)	Maximise $P = 12x + 20y + 16z$	B1	3.3
	Subject to $2x+3y+z \le 80$ $4x+2y+3z \le 140$ $3x+4y+2z \le 96$ $x, y, z \ge 0$	M1 A1 A1 B1	3.3 1.1b 1.1b 3.3
		(5)	
(b)	The values must all be integers	B1	3.3
		(1)	
(c)	Variable y entered the basic variable column…	M1	2.4
	…so y was increased first	A1	2.2a
		(2)	
(d)	$(80+140+96)-(8+92)=216$ plants	B1	3.2a
		(1)	
(e)	The next pivot must come from a column which has a negative value in the objective row so therefore the pivot must come from column z	M1	2.4
	The pivot must be positive and the least of 92/2 = 46 and 24/0.5 = 48 so the pivot must be the 2 (from column z)	A1	2.2a
		(2)	
(f)	P + 10.5x + 3s + 3.5t = 756 so increasing x, s or t will decrease profit	B1	2.4
		(1)	
(g)	Make 1 *Drama* basket and 46 *Peaceful* baskets	B1	2.2a
		(1)	
(h)	The slack variable, r, associated with this type of plant, is currently at 31. Increasing the number of *Impact* plants by a further 20 would have no effect	M1 A1	3.1b 3.2a
		(2)	
		(15 marks)	

Question 5 notes:
(a) **B1:** Correct objective function/expression (accept in pence rather than pounds e.g. $1200x + 2000y + 1600z$) **M1:** Correct coefficients and correct right-hand side for at least one inequality – accept any inequality or equals **A1:** Two correct (non-trivial) inequalities **A1:** All three non-trivial inequalities correct **B1:** $x, y, z \geq 0$
(b) **B1:** cao
(c) **M1:** Correct reasoning that y has become a basic variable **A1:** Correct deduction that y was therefore increased first
(d) **B1**: cao
(e) **M1:** Correct reasoning given that the pivot value must come from column z **A1:** Correctly deduce (from correctly stated calculations) that the pivot value is the 2 in column z
(f) **B1:** States correct objective function and mention of increasing x, s or t will decrease profit
(g) **B1:** cao – in context so not in terms of y and z
(h) **M1:** Identifies the slack variable r and its current value of 31 **A1:** Correct interpretation that increasing the number of Impact plants would have no effect

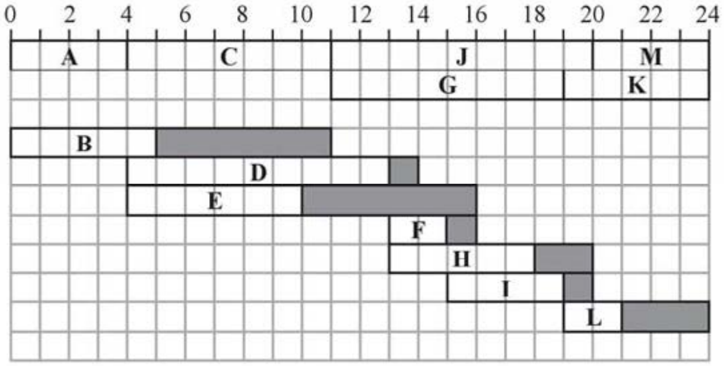
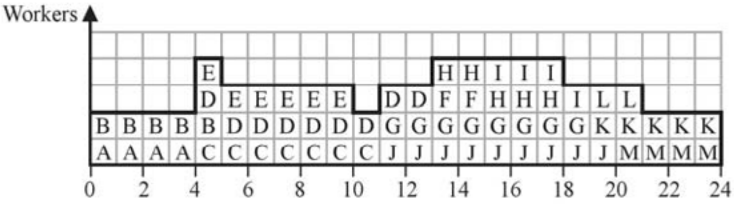
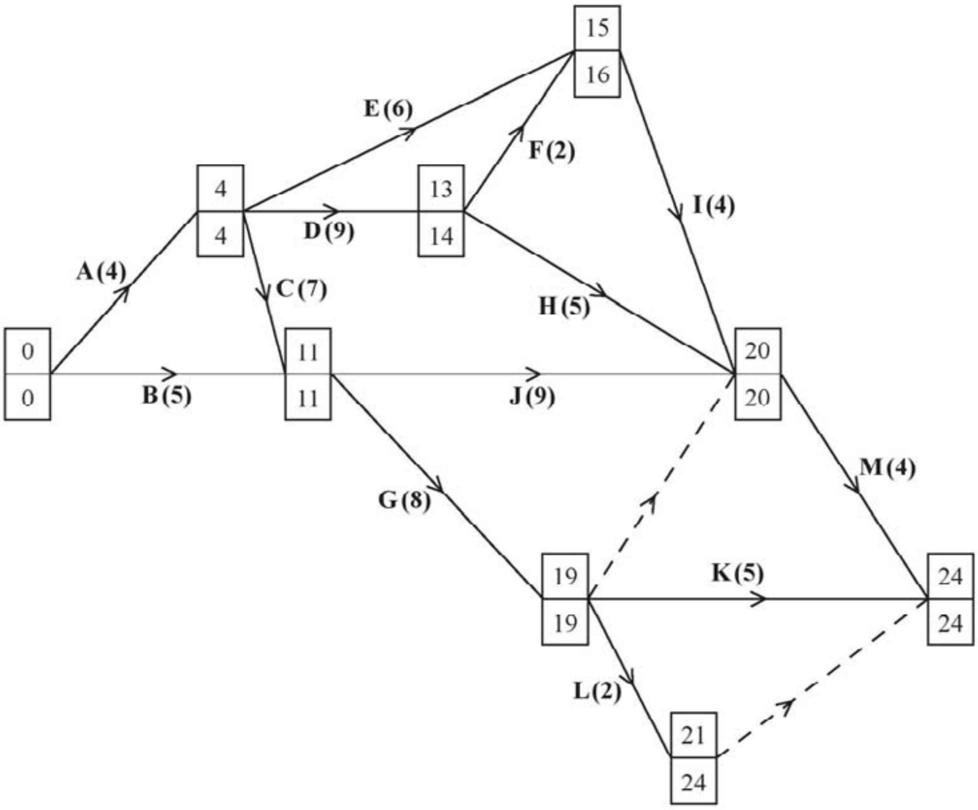
Question	Scheme	Marks	AOs
6(a)	**See diagram on next page.** Top and bottom boxes Top boxes correct	M1 A1	2.1 1.1b
	Bottom boxes correct	A1	1.1b
		(3)	
(b)	See diagram below 		
	At least 8 activities + 4 floats with clear distinction between activity and their corresponding float	M1	2.5
	Correct critical activities + 4 correct non-critical activities	A1	1.1b
	All 13 correct	A1	1.1b
		(3)	
(c)		M1	1.1b
	Bars correct to time = 13	A1	1.1b
	Bars correct from 14 to 24	A1	1.1b
		(3)	
(d)	Until time 4 only A and B can happen. After time 4, there are 6 worker-days to cover, but only 4 worker-days available. Hence the project cannot be completed by time 24 with three workers.	B1 M1 A1	3.1a 2.4 2.2a
		(3)	
		(12 marks)	

Diagram for Question 6(a)



Question 6 notes:

(a)

M1: All top boxes and all bottom boxes completed. For the top boxes all values must be increasing in the direction of the arrows for both the activities and the dummies. For the bottom boxes all values must be decreasing in the opposite direction to the arrows for both the activities and the dummies. While the values need not be correct each value must be increasing or decreasing (as appropriate) in a logical and sequential manner.

A1: cao for top boxes

A1: cao for bottom boxes

M1: At least 8 activities including 4 floats. Scheduling diagram scores M0 – clear distinction must be shown between the notation used for an activity and its float

(b)

A1: Correct critical activities and 4 correct non-critical activities

A1: cao (all 13 correct activities)

(c)

M1: Plausible histogram with no holes or overhangs (must go to at least 10 on the time axis)

A1: Histogram correct to time 13

A1: Histogram correct from time 14 to time 24

(d)

B1: Considering an appropriate process to adjust Grid 2 so that no activity must be completed by a 4th worker, for example, a correct argument that until time 4 only activities A and B can happen (so no activity can use the spare worker before time 4)

M1: Uses their histogram to explain when the number of workers is greater or less than the minimum number found in (b)

A1: Correctly deduces that the project cannot be completed by time 24 – this mark is dependent on a correct histogram seen in (d)

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td>7(a)</td><td>Simplex can only work with ≤ constraints</td><td>B1</td><td>3.5b</td></tr>
<tr><td></td><td></td><td>(1)</td><td></td></tr>
<tr><td>(b)</td><td>M is an arbitrary large real number</td><td>B1</td><td>2.5</td></tr>
<tr><td></td><td></td><td>(1)</td><td></td></tr>
<tr><td>(c)</td><td>$x \geqslant 3 \Rightarrow x - s_3 + t_1 = 3$ where s_3 is a surplus variable and t_1 is an artificial variable</td><td>B1</td><td>2.4</td></tr>
<tr><td></td><td></td><td>(1)</td><td></td></tr>
<tr><td>(d)</td><td>Let $P = 3x + 2y + 2z - Mt_1$ (where M is an arbitrary large number)
$\therefore P = 3x + 2y + 2z - M(3 - x + s_3)$
$= (3 + M)x + 2y + 2z - Ms_3 - 3M$
$\Rightarrow P - (3 + M)x - 2y - 2z + Ms_3 = -3M$</td><td>M1
A1</td><td>2.1
1.1b</td></tr>
<tr><td></td><td></td><td>(2)</td><td></td></tr>
<tr><td>(e)</td><td>
<table>
<tr><th>b.v.</th><th>x</th><th>y</th><th>z</th><th>s_1</th><th>s_2</th><th>s_3</th><th>t_1</th><th>Value</th><th>Row Ops</th></tr>
<tr><td>s_3</td><td>0</td><td>1</td><td>½</td><td>½</td><td>0</td><td>1</td><td>−1</td><td>19/2</td><td>$r_1 = (1/2)R_1$</td></tr>
<tr><td>s_2</td><td>0</td><td>3</td><td>−1/2</td><td>−1/2</td><td>1</td><td>0</td><td>0</td><td>5/2</td><td>$R_2 - r_1$</td></tr>
<tr><td>x</td><td>1</td><td>1</td><td>½</td><td>½</td><td>0</td><td>0</td><td>0</td><td>25/2</td><td>$R_3 + r_1$</td></tr>
<tr><td>P</td><td>0</td><td>1</td><td>−1/2</td><td>3/2</td><td>0</td><td>0</td><td>M</td><td>75/2</td><td>$R_4 + 3r_1$</td></tr>
</table>
</td><td>M1
A1
A1</td><td>1.1b
1.1b
1.1b</td></tr>
<tr><td></td><td>
<table>
<tr><th>b.v.</th><th>x</th><th>y</th><th>z</th><th>s_1</th><th>s_2</th><th>s_3</th><th>t_1</th><th>Value</th><th>Row Ops</th></tr>
<tr><td>z</td><td>0</td><td>2</td><td>1</td><td>1</td><td>0</td><td>2</td><td>−2</td><td>19</td><td>$r_1 = 2R_1$</td></tr>
<tr><td>s_2</td><td>0</td><td>4</td><td>0</td><td>0</td><td>1</td><td>1</td><td>−1</td><td>12</td><td>$R_2 + (1/2)r_1$</td></tr>
<tr><td>x</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>−1</td><td>1</td><td>3</td><td>$R_3 - (1/2)r_1$</td></tr>
<tr><td>P</td><td>0</td><td>2</td><td>0</td><td>2</td><td>0</td><td>1</td><td>$M - 1$</td><td>47</td><td>$R_4 + (1/2)r_1$</td></tr>
</table>
</td><td>M1
A1
B1</td><td>1.1b
1.1b
2.4</td></tr>
<tr><td></td><td>$P = 47, x = 3, y = 0, z = 19$</td><td>B1ft</td><td>1.1b</td></tr>
<tr><td></td><td></td><td>(7)</td><td></td></tr>
<tr><td></td><td colspan="3">(12 marks)</td></tr>
</table>

Question 7 notes:

(a)

B1: Correctly states the limitation of the Simplex model – Simplex involves iterations which allow movement from one vertex in the feasible region to another vertex (in the feasible region). If all constraints are of the form $\leq$ this means that the origin is always a feasible solution and therefore can act as the initial starting point for the problem. However, the constraint $x \geqslant 3$ means that the origin is not feasible and so the algorithm is unable to begin.

(b)

B1: cao including the correct mathematical language (must include 'arbitrary', 'large' and 'real')

(c)

(B1: Correctly states both the inequality $x \geq 3$ and the equation $x - s_3 + t_1 = 3$ together with an explanation of the meaning behind the variables s_3 and t_1

(d)

M1: $P = 3x + 2y + 2z - Mt_1$ and substitutes their expression for t_1

A1: Correct mathematical argument including sufficient detail to allow the line of reasoning to be followed to the correct conclusion – dependent on previous B mark in (c)

(e)

M1: Correct pivot located, attempt to divide row. If negative value used then no marks

A1: Pivot row correct (including change of b.v.) and row operations used at least once, one of columns y, z, s_1, t_1 or Value correct

A1: cao for values (ignore b.v. column and Row Ops)

M1: Pivot row consistent (following their previous table) including change of b.v. and row operations used at least once, one of columns y, s_1, s_3, t_1 or Value correct

A1: cao on final table (ignore Row Ops)

B1: The correct Row Operations explained either in terms of the 'old' or 'new' pivot rows

B1ft: Correctly states the final values of P, x, y and z from their correct corresponding rows of the final table

Pearson Edexcel Level 3 GCE

Further Mathematics

Advanced
Further Mathematics Option 2
Paper 4: Decision Mathematics 2

Sample Assessment Material for first teaching September 2017
Time: 1 hour 30 minutes

Paper Reference
9FM0/4G

You must have:
Decision Mathematics Answer Book (enclosed), calculator

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Write your answers for this paper in the Decision Mathematics answer book provided.
- **Fill in the boxes** at the top of the answer book with your name, centre number and candidate number.
- Do not return the question paper with the answer book.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read **each** question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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1/1/1/1/1/

S 5 4 4 4 7 A

Answer ALL questions. Write your answers in the answer book provided.

1. (a) Find the general solution of the recurrence relation

$$u_{n+2} = u_{n+1} + u_n, \quad n \geqslant 1$$

(3)

Given that $u_1 = 1$ and $u_2 = 1$

(b) find the particular solution of the recurrence relation.

(3)

(Total for Question 1 is 6 marks)

2.

	D	E	F	Available
A	15	19	9	25
B	11	18	10	55
C	11	12	18	20
Required	38	24	38	

A company has three factories, A, B and C. It supplies mattresses to three shops, D, E and F. The table shows the transportation cost, in pounds, of moving one mattress from each factory to each shop. It also shows the number of mattresses available at each factory and the number of mattresses required at each shop. A minimum cost solution is required.

(a) Use the north-west corner method to obtain an initial solution.

(1)

(b) Show how the transportation algorithm is used to solve this problem.

You must state, at each appropriate step, the

- shadow costs,
- improvement indices,
- route,
- entering cell and exiting cell,

and explain clearly how you know that your final solution is optimal.

(11)

(Total for Question 2 is 12 marks)

3. Four workers, A, B, C and D, are to be assigned to four tasks, P, Q, R and S.

Each worker must be assigned to at most one task and each task must be done by just one worker.

The amount, in pounds, that each worker would earn while assigned to each task is shown in the table below.

	P	Q	R	S
A	32	32	33	35
B	28	35	31	37
C	35	29	33	36
D	36	30	36	33

The Hungarian algorithm is to be used to find the maximum total amount which may be earned by the four workers.

(a) Explain how the table should be modified.

(1)

(b) Reducing rows first, use the Hungarian algorithm to obtain an allocation which maximises the total earnings, stating how each table was formed.

(7)

(c) Formulate the problem as a linear programming problem. You must define your decision variables and make your objective function and constraints clear.

(5)

(Total for Question 3 is 13 marks)

4. A game uses a standard pack of 52 playing cards.

A player gives 5 tokens to play and then picks a card. If they pick a 2, 3, 4, 5 or 6 then they gain 15 tokens. If any other card is picked they lose.

If they lose, the card is replaced and they can choose to pick again for another 5 tokens. This time if they pick either an ace or a king they gain 40 tokens. If any other card is picked they lose.

Daniel is deciding whether to play this game.

(a) Draw a decision tree to model Daniel's possible decisions and the possible outcomes.

(6)

(b) Calculate Daniel's optimal EMV and state the optimal strategy indicated by the decision tree.

(2)

(Total for Question 4 is 8 marks)

5.

	B plays 1	B plays 2	B plays 3	B plays 4
A plays 1	4	−2	3	2
A plays 2	3	−1	2	0
A plays 3	−1	2	0	3

A two person zero-sum game is represented by the pay-off matrix for player A given above.

(a) Explain, with justification, how this matrix may be reduced to a 3 × 3 matrix.

(2)

(b) Find the play-safe strategy for each player and verify that there is no stable solution to this game.

(4)

The game is formulated as a linear programming problem for player A.

The objective is to maximise $P = V$, where V is the value of the game to player A.

One of the constraints is that $p_1 + p_2 + p_3 \leqslant 1$, where p_1, p_2, p_3 are the probabilities that player A plays 1, 2, 3 respectively.

(c) Formulate the remaining constraints for this problem. Write these constraints as inequalities.

(3)

The Simplex algorithm is used to solve the linear programming problem.

The solution obtained is $p_1 = 0, p_2 = \frac{3}{7}, p_3 = \frac{4}{7}$

(d) Calculate the value of the game to player A.

(3)

(Total for Question 5 is 12 marks)

6.

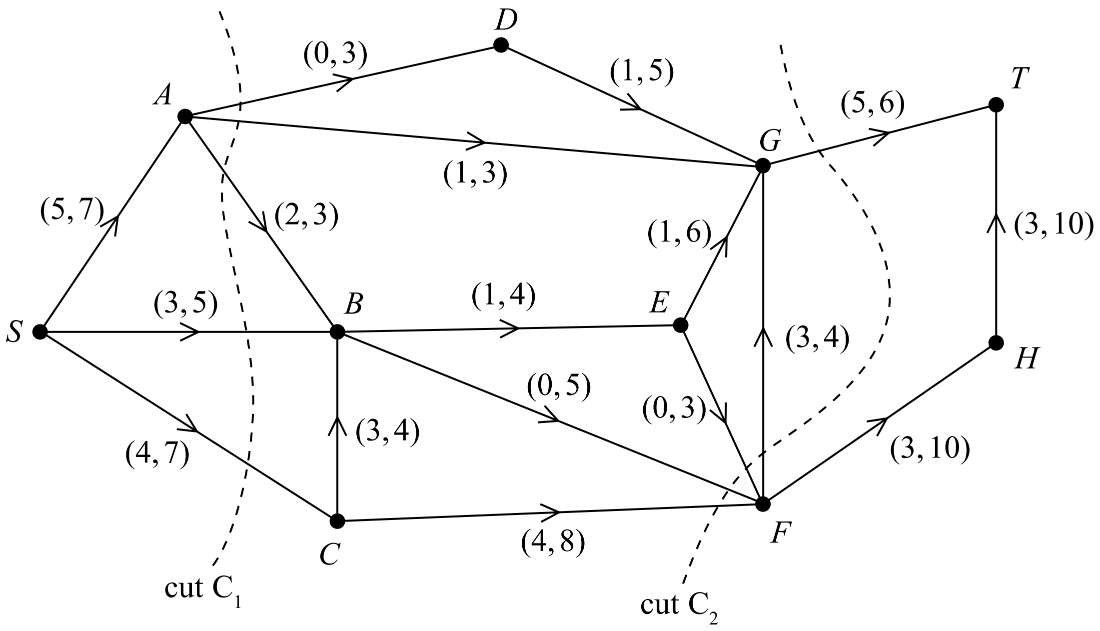


Figure 1

Figure 1 shows a capacitated, directed network. The number on each arc (*x*, *y*) represents the lower (*x*) capacity and upper (*y*) capacity of that arc.

(a) Calculate the value of the cut C_1 and cut C_2

(2)

(b) Explain why the flow through the network must be at least 12 and at most 16

(1)

(c) Explain why arcs DG, AG, EG and FG must all be at their lower capacities.

(1)

(d) Determine a maximum flow pattern for this network and draw it on Diagram 1 in the answer book. You do not need to use the labelling procedure.

(2)

(e) (i) State the value of the maximum flow through the network.

(ii) Explain why the value of the maximum flow is equal to the value of the minimum flow through the network.

(3)

Node E becomes blocked and no flow can pass through it. To maintain the maximum flow through the network the upper capacity of exactly one arc is increased.

(f) Explain how it is possible to maintain the maximum flow found in (d).

(3)

(Total for Question 6 is 12 marks)

7. A company assembles boats.

They can assemble up to five boats in any one month, but if they assemble more than three they will have to hire additional space at a cost of £800 per month.

The company can store up to two boats at a cost of £350 each per month.

The overhead costs are £1500 in any month in which work is done.

Boats are delivered at the end of each month. There are no boats in stock at the beginning of January and there must be none in stock at the end of May.

The order book for boats is

Month	January	February	March	April	May
Number ordered	3	2	6	3	4

Use dynamic programming to determine the production schedule which minimises the costs to the company. Show your working in the table provided in the answer book and state the minimum production cost.

(Total for Question 7 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS

Write your name here

Surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number ☐☐☐☐☐

Candidate Number ☐☐☐☐

Further Mathematics

Advanced
Further Mathematics Option 2
Paper 4: Decision Mathematics 2

Sample Assessment Material for first teaching September 2017

Paper Reference
9FM0/4G

Answer Book
Do not return the question paper with the answer book.

Total Marks

Turn over ▶

S54447A

1/1/1/1/1/

S 5 4 4 4 7 A 0 1 1 6

1.

Question 1 continued

(Total for Question 1 is 6 marks)

2.

	D	E	F	Available
A	15	19	9	25
B	11	18	10	55
C	11	12	18	20
Required	38	24	38	

(a)

	D	E	F	
A				
B				
C				

(b)

	D	E	F	
A				
B				
C				

	D	E	F	
A				
B				
C				

	D	E	F	
A				
B				
C				

	D	E	F	
A				
B				
C				

	D	E	F	
A				
B				
C				

Question 2 continued

	D	E	F	
A				
B				
C				

	D	E	F	
A				
B				
C				

	D	E	F	
A				
B				
C				

	D	E	F	
A				
B				
C				

	D	E	F	
A				
B				
C				

	D	E	F	
A				
B				
C				

	D	E	F	
A				
B				
C				

(Total for Question 2 is 12 marks)

3.

	P	Q	R	S
A	32	32	33	35
B	28	35	31	37
C	35	29	33	36
D	36	30	36	33

	P	Q	R	S
A				
B				
C				
D				

	P	Q	R	S
A				
B				
C				
D				

	P	Q	R	S
A				
B				
C				
D				

	P	Q	R	S
A				
B				
C				
D				

	P	Q	R	S
A				
B				
C				
D				

Question 3 continued

	P	Q	R	S
A				
B				
C				
D				

	P	Q	R	S
A				
B				
C				
D				

(Total for Question 3 is 13 marks)

4.

Question 4 continued

(Total for Question 4 is 8 marks)

5.

	B plays 1	B plays 2	B plays 3	B plays 4
A plays 1	4	–2	3	2
A plays 2	3	–1	2	0
A plays 3	–1	2	0	3

Question 5 continued

(Total for Question 5 is 12 marks)

6.

Question 6 continued

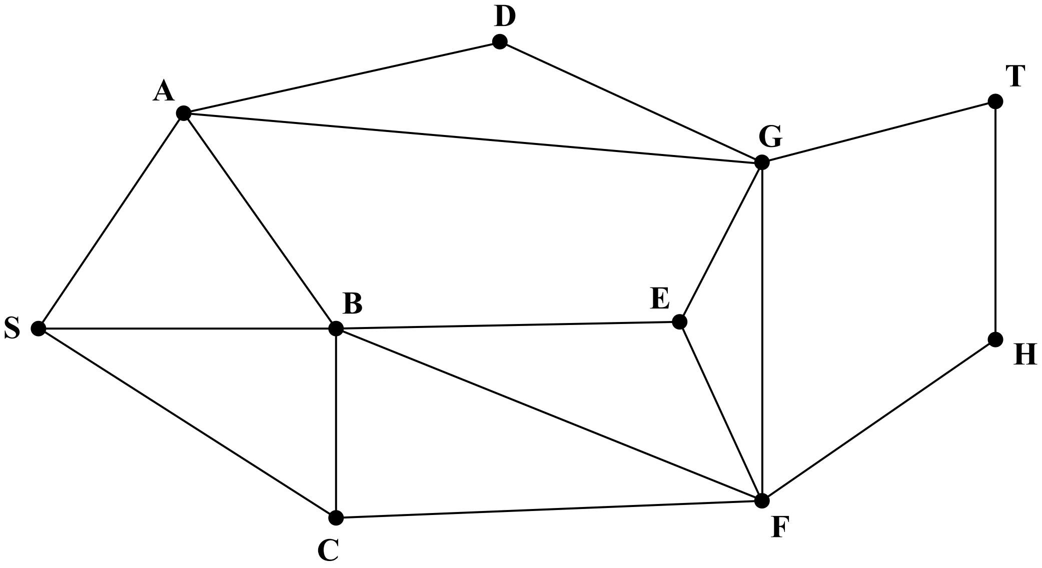


Diagram 1

(Total for Question 6 is 12 marks)

7.

Stage	State	Action	Dest	Value

Month	January	February	March	April	May
Number assembled					

Question 7 continued

(Total for Question 7 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS

Paper 4G: Decision Mathematics 2 Mark Scheme

Question	Scheme	Marks	AOs
1(a)	Auxiliary equation: $\lambda^2 - \lambda - 1 = 0$ and attempt to solve	M1	1.1b
	$\lambda = \frac{1 \pm \sqrt{5}}{2} \Rightarrow u_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$, where A and B are arbitrary constants	M1 A1	1.1b 2.2a
		(3)	
(b)	Use given conditions to obtain two equations in A and B	M1	1.1b
	Attempt to solve to obtain an A and B	M1	1.1b
	$u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$	A1	1.1b
		(3)	
		(6 marks)	

Notes:

(a)

M1: writes down correct auxiliary equation and attempts to solve using either the formula or completing the square

M1: writes down the general solution in the form $u_n = A(\lambda_1)^n + B(\lambda_2)^n$ using their roots λ_1, λ_2 - dependent on the first M mark

A1: CAO – both lhs and rhs correct including defining *A* and *B* as (arbitrary) constants

(b)

M1: uses the correct initial conditions to write down two equations in *A* and *B* – for reference these equations are $A\left(1+\sqrt{5}\right)+B\left(1-\sqrt{5}\right)=2$ and $A\left(1+\sqrt{5}\right)^2+B\left(1-\sqrt{5}\right)^2=4$

M1: Attempts to solve these two equations (using a correct method but condone sign slips) to achieve a value for *A* and *B*

A1: CAO

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td rowspan="2">2(a)</td><td>
<table>
<tr><td></td><td>D</td><td>E</td><td>F</td><td>Available</td></tr>
<tr><td>A</td><td>25</td><td></td><td></td><td>25</td></tr>
<tr><td>B</td><td>13</td><td>24</td><td>18</td><td>55</td></tr>
<tr><td>C</td><td></td><td></td><td>20</td><td>20</td></tr>
<tr><td>Required</td><td>38</td><td>24</td><td>38</td><td></td></tr>
</table>
</td><td>B1</td><td>1.1b</td></tr>
<tr><td></td><td>(1)</td><td></td></tr>
<tr><td rowspan="6">(b)</td><td>
<table>
<tr><td>Shadow costs</td><td></td><td>15</td><td>22</td><td>14</td></tr>
<tr><td></td><td></td><td>D</td><td>E</td><td>F</td></tr>
<tr><td>0</td><td>A</td><td>X</td><td>-3</td><td>-5</td></tr>
<tr><td>-4</td><td>B</td><td>X</td><td>X</td><td>X</td></tr>
<tr><td>4</td><td>C</td><td>-8</td><td>-14</td><td>X</td></tr>
</table>
</td><td>M1
A1</td><td>2.1
1.1b</td></tr>
<tr><td>
<table>
<tr><td></td><td>D</td><td>E</td><td>F</td></tr>
<tr><td>A</td><td></td><td></td><td></td></tr>
<tr><td>B</td><td></td><td>$24-\theta$</td><td>$18+\theta$</td></tr>
<tr><td>C</td><td></td><td>θ</td><td>$20-\theta$</td></tr>
</table>
Entering CE, exiting CF
<table>
<tr><td></td><td>D</td><td>E</td><td>F</td></tr>
<tr><td>A</td><td>25</td><td></td><td></td></tr>
<tr><td>B</td><td>13</td><td>4</td><td>38</td></tr>
<tr><td>C</td><td></td><td>20</td><td></td></tr>
</table>
</td><td>M1
A1</td><td>2.1
1.1b</td></tr>
<tr><td>
<table>
<tr><td>Shadow costs</td><td></td><td>15</td><td>22</td><td>14</td></tr>
<tr><td></td><td></td><td>D</td><td>E</td><td>F</td></tr>
<tr><td>0</td><td>A</td><td>X</td><td>-3</td><td>-5</td></tr>
<tr><td>-4</td><td>B</td><td>X</td><td>X</td><td>X</td></tr>
<tr><td>-10</td><td>C</td><td>6</td><td>X</td><td>14</td></tr>
</table>
</td><td>M1
A1</td><td>1.1b
1.1b</td></tr>
<tr><td>
<table>
<tr><td></td><td>D</td><td>E</td><td>F</td></tr>
<tr><td>A</td><td>$25-\theta$</td><td></td><td>θ</td></tr>
<tr><td>B</td><td>$13+\theta$</td><td></td><td>$38-\theta$</td></tr>
<tr><td>C</td><td></td><td></td><td></td></tr>
</table>
<table>
<tr><td></td><td>D</td><td>E</td><td>F</td></tr>
<tr><td>A</td><td></td><td></td><td>25</td></tr>
<tr><td>B</td><td>38</td><td>4</td><td>13</td></tr>
<tr><td>C</td><td></td><td>20</td><td></td></tr>
</table>
Entering AF, exiting AD
</td><td>M1
A1</td><td>1.1b
2.2a</td></tr>
<tr><td>
<table>
<tr><td>Shadow costs</td><td></td><td>10</td><td>17</td><td>9</td></tr>
<tr><td></td><td></td><td>D</td><td>E</td><td>F</td></tr>
<tr><td>0</td><td>A</td><td>5</td><td>2</td><td>X</td></tr>
<tr><td>1</td><td>B</td><td>X</td><td>X</td><td>X</td></tr>
<tr><td>-5</td><td>C</td><td>6</td><td>X</td><td>14</td></tr>
</table>
No negative IIs so optimal solution of £1085
</td><td>M1
A1
A1</td><td>2.1
1.1b
2.4</td></tr>
<tr><td></td><td>(11)</td><td></td></tr>
<tr><td colspan="4">(12 marks)</td></tr>
</table>

Question 2 notes:

B1:	CAO
M1:	Finding all 6 shadow costs and the 4 improvement indices for the correct 4 entries – candidates must clearly identify these two sets of results
A1:	Shadow costs and II CAO
M1:	A valid route, their most negative II chosen, only one empty square used, θ's balance
A1:	CAO
M1:	Finding all 6 shadow costs and the 4 improvement indices for the correct 4 entries
A1:	Shadow costs and II CAO
M1:	A valid route, their most negative II chosen, only one empty square used, θ's balance
A1:	CAO – including the deduction of all entering and exiting cells
M1:	Finding all 6 shadow costs and the 4 improvement indices for the correct 4 entries – this mark is depedent on all previous M marks which will therefore indicate a correct mathematical argument leading from the initial solution to the confirmation of the optimal solution
A1:	Shadow costs and II CAO
A1:	CSO including the correct reasoning that the solution is optimal because there are no negative IIs

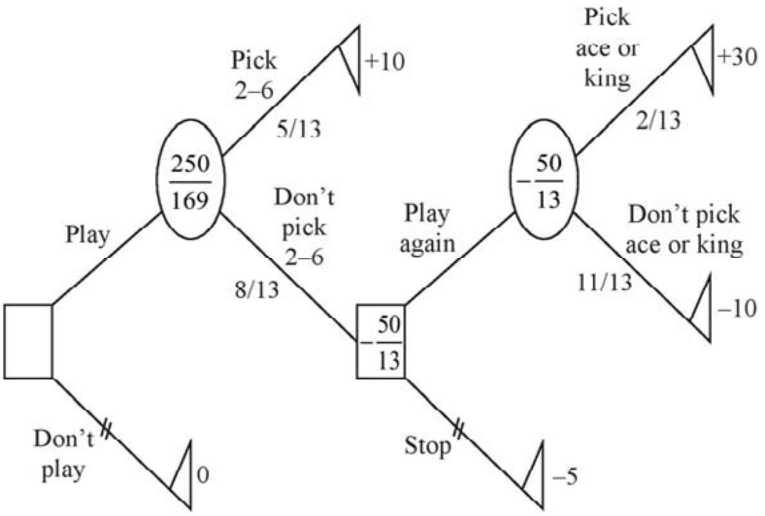
Question	Scheme	Marks	AOs
3(a)	Subtract each entry from a constant (eg 40)	B1	2.4
		(1)	
(b)	e.g. $\left(\begin{array}{ccccc} & \text{P} & \text{Q} & \text{R} & \text{S} \\ \text{A} & 5 & 5 & 4 & 2 \\ \text{B} & 9 & 2 & 6 & 0 \\ \text{C} & 2 & 8 & 4 & 1 \\ \text{D} & 1 & 7 & 1 & 4 \end{array}\right)$	B1	1.1b
	Reducing row A by 2, no reduction for row B, reduce row C by 1 and row D by 1. No reduction of columns P, R and S, reduce column Q by 2. $\left(\begin{array}{ccccc} & \text{P} & \text{Q} & \text{R} & \text{S} \\ \text{A} & 3 & 3 & 2 & 0 \\ \text{B} & 9 & 2 & 6 & 0 \\ \text{C} & 1 & 7 & 3 & 0 \\ \text{D} & 0 & 6 & 0 & 3 \end{array}\right)$ then $\left(\begin{array}{ccccc} & \text{P} & \text{Q} & \text{R} & \text{S} \\ \text{A} & 3 & 1 & 2 & 0 \\ \text{B} & 9 & 0 & 6 & 0 \\ \text{C} & 1 & 5 & 3 & 0 \\ \text{D} & 0 & 4 & 0 & 3 \end{array}\right)$	B1 M1 A1	2.4 2.1 1.1b
	Three lines required to cover the zeros hence solution is not optimal – augment by 1	B1	2.4
	$\left(\begin{array}{ccccc} & \text{P} & \text{Q} & \text{R} & \text{S} \\ \text{A} & 2 & 1 & 1 & 0 \\ \text{B} & 8 & 0 & 5 & 0 \\ \text{C} & 0 & 5 & 2 & 0 \\ \text{D} & 0 & 5 & 0 & 4 \end{array}\right)$	M1	2.1
	A – S, B – Q, C – P, D – R	A1	2.2a
		(7)	
(c)	$x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \qquad\qquad \text{otherwise} \end{cases}$	B1	3.3
	Where $i \in \{\text{A,B,C,D}\}$ and $j \in \{\text{P,Q,R,S}\}$	B1	3.3
	e.g. Minimise $5x_{\text{AP}} + 5x_{\text{AQ}} + 4x_{\text{AR}} + 2x_{\text{AS}} + 9x_{\text{BP}} + 2x_{\text{BQ}} + 6x_{\text{BR}} +$		
	$2x_{\text{CP}} + 8x_{\text{CQ}} + 4x_{\text{CR}} + x_{\text{CS}} + x_{\text{DP}} + 7x_{\text{DQ}} + x_{\text{DR}} + 4x_{\text{DS}}$	B1	3.3
	Subject to:		
	$\sum x_{i\text{P}} = 1, \sum x_{i\text{Q}} = 1, \sum x_{i\text{R}} = 1, \sum x_{i\text{S}} = 1$	M1	3.3
	$\sum x_{\text{A}j} = 1, \sum x_{\text{B}j} = 1, \sum x_{\text{C}j} = 1, \sum x_{\text{D}j} = 1$	A1	3.3
		(5)	
		(13 marks)	

Question 3 notes:

(a)
B1: valid statement regarding converting a max. problem to a min. problem

(b)
B1: CAO
B1: Correct statements regarding row and column reduction
M1: Simplifying the initial matrix by reducing rows **and then** columns
A1: CAO
B1: Correct statements regarding both max. number of lines to cover zeros and augmentation
M1: Develop an improved solution – need to see one double covered +e; one uncovered –e; and one single covered unchanged. 3 lines needed to 4 lines needed (so getting to the optimal table)
A1: CSO on final table (so must have scored all previous marks in this part) + deduction of the correct allocation

(c)
B1: possible values of x_{ij} defined
B1: definine the set of values for i and j
B1: Correct objective function and either 'minimise' or 'maximise' (dependent on if problem is defined in terms of original values or modified values)
M1: at least four equations, unit coefficient and equal to 1
A1: CAO (all eight equations)

Question	Scheme	Marks	AOs
4(a)		M1 A1 M1 A1 M1 A1	3.3 1.1b 3.4 1.1b 3.4 1.1b
		(6)	
(b)	EMV is 1.48 (tokens) per game (correct to 3 sf) Analysis: Play the game and if the player doesn't pick a 2 – 6 on the first go then they should pick again	B1 B1	3.4 3.2a
		(2)	
			(8 marks)

Notes:

(a)

M1: Tree diagram with at least three end pay-offs, two decision nodes and two chance nodes

A1: Correct structure of tree diagram with each arc labelled correctly (including probabilities)

M1: At least three end-pay offs consistent with their stated probabilities; all five attempted

A1: CAO for end-pay offs

M1: End chance node follow through their end pay-offs and other chance/decision nodes completed

A1: CAO for decision and chance nodes including double lines through inferior options

(b)

B1: Correct EMV

B1: Correct analysis

Question	Scheme	Marks	AOs
5(a)	Column 2 dominates column 4	B1	2.5
	Because $2 > -2, 0 > -1$ and $3 > 2$	B1	2.4
		(2)	
(b)	Row minima: -2, -1, -1 max is -1 Column maxima: 4, 2, 3 min is 2	M1 A1	1.1b 1.1b
	Play safe is A plays 2 or 3 and B plays 2	A1	1.1b
	Row maximin (-1) $\neq$ Column minimax (2) so not stable	A1	2.4
		(4)	
(c)	e.g. $\begin{pmatrix} 4 & -2 & 3 \\ 3 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 6 & 0 & 5 \\ 5 & 1 & 4 \\ 1 & 4 & 2 \end{pmatrix}$ Subject to $V - 6p_1 - 5p_2 - p_3 \leqslant 0$ $V - p_2 - 4p_3 \leqslant 0$ $V - 5p_1 - 4p_2 - 2p_3 \leqslant 0$	B1 B1 B1	1.1b 3.3 3.3
		(3)	
(d)	Substitute p values to obtain $V \leq \frac{19}{7}, \frac{19}{7}, \frac{20}{7} \therefore V = \frac{19}{7}$	M1	3.4
	Value of the game to player A $= \frac{19}{7} - 2 = \frac{5}{7}$	M1 A1	1.1b 1.1b
		(3)	
		(12 marks)	

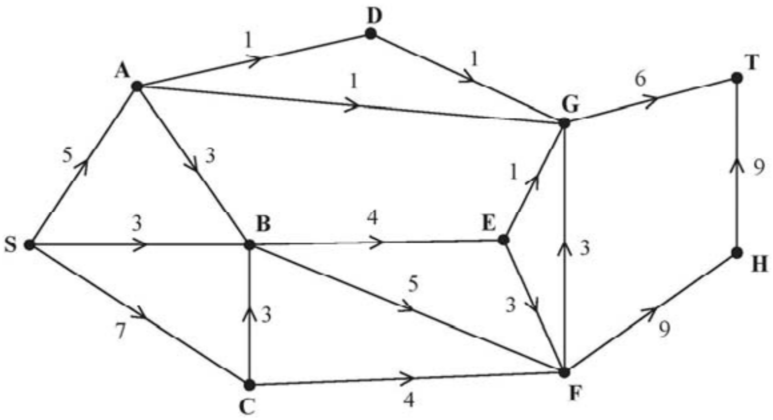
Notes:

(a)
B1: Correct statement – must include the word 'dominate'
B1: Correct inequalities – must be clear that all three inequalities must hold

(b)
M1: Attempt at row minima and column maxima – condone one error
A1: Correct max(row min) and min(col max)
A1: Correct play safe for both players
A1: Correct reasoning that the game is not stable (accept $-1 \neq 2$ + statement)

(c)
B1: Correct augmentation to make all entries non-negative
B1: At least one (of the three) equations **or** inequalities correct in V, p_1, p_2, p_3 (with all p_i terms in the constraint equations having correct signs)
B1: CAO - all three constraints correct involving V and p_i expressed as inequalities

(d)
M1: Substitute p values to obtain three values for V
M1: Their least value of V minus their augmented value
A1: CAO for the value of the game to player A

Question	Scheme	Marks	AOs
6(a)	$C_1 = 3+3+3+5+7=21$ $C_2 = 8+5+3+6-3=19$	B1 B1	1.1b 1.1b
		(2)	
(b)	e.g. the minimum flow out of the source S is at least 5 + 3 + 4 = 12 and the maximum flow into the sink T is 6 + 10 = 16	B1	2.4
		(1)	
(c)	The minimum flow into G is 1 + 1 + 1 + 3 = 6 but the maximum flow out of G is 6 therefore the arcs into G must be at their lower capacities	B1	2.4
		(1)	
(d)		M1 A1	3.1a 1.1b
		(2)	
(e)	Maximum flow is 15	B1	1.1b
	The minimum flow out of the source is 12 but the flow out of C is at least 3 + 4 = 7	B1	2.4
	Therefore the minimum flow through the network is 5 + 3 + 3 + 4 = 15 which is equal to the maximum flow	B1	2.2a
		(3)	
(f)	Increase the upper capacity of arc BF to at least 9 and therefore increase the flow in this arc to 9	B1	2.1
	Therefore increase the flow in FH and HT to 10	B1	2.4
	The flow in GT decreases to 5 and all other arcs are unchanged	B1	2.2a
		(3)	
		(12 marks)	

Question 6 notes:
(a) **B1:** correct capacity for C_1 **B1:** correct capacity for C_2
(b) **B1:** correct statement regarding the min. flow out of the sink and max. flow into the sink
(c) **B1:** correct statement regarding the flow into node G
(d) **M1:** consistent flow pattern (≥ 12) throughout the network **-** so the flow into each node must equal the flow out of each node (and this flow must be greater than or equal to 12 but not necessarily the maximum flow of 15) - one number only on each arc **A1:** CAO
(e) **B1:** CAO (for max. flow) **B1:** Consideration of both the min. flow from the source and the flow through node C **B1:** Completely correct argument that the max. flow = min. flow
(f) **B1:** Correct argument regarding increasing the upper capacity of arc BF and hence the flow in that arc **B1:** Correct reasoning regarding increasing the flow in arcs FH and HT **B1:** Correct deduction that the flow in GT decreases to 5 and conclude that all other arcs are unchanged

<table>
<tr><th>Question</th><th>Scheme</th><th>Marks</th><th>AOs</th></tr>
<tr><td>7</td><td>
<table>
<tr><th>Stage</th><th>State</th><th>Action</th><th>Dest</th><th>Value</th></tr>
<tr><td>May</td><td>2</td><td>2</td><td>0</td><td>700 + 1500 = 2200*</td></tr>
<tr><td>(4)</td><td>1</td><td>3</td><td>0</td><td>350 + 1500 = 1850*</td></tr>
<tr><td></td><td>0</td><td>4</td><td>0</td><td>1500 + 800 = 2300*</td></tr>
<tr><td>April</td><td>2</td><td>1</td><td>0</td><td>700 + 1500 + 2300 = 4500</td></tr>
<tr><td>(3)</td><td></td><td>2</td><td>1</td><td>700 + 1500 + 1850 = 4050*</td></tr>
<tr><td></td><td></td><td>3</td><td>2</td><td>700 + 1500 + 2200 = 4400</td></tr>
<tr><td></td><td>1</td><td>2</td><td>0</td><td>350 + 1500 + 2300 = 4150</td></tr>
<tr><td></td><td></td><td>3</td><td>1</td><td>350 + 1500 + 1850 = 3700*</td></tr>
<tr><td></td><td></td><td>4</td><td>2</td><td>350 + 1500 + 800 +2200 = 4850</td></tr>
<tr><td></td><td>0</td><td>3</td><td>0</td><td>1500 + 2300 = 3800*</td></tr>
<tr><td></td><td></td><td>4</td><td>1</td><td>1500 + 800 + 1850 = 4150</td></tr>
<tr><td></td><td></td><td>5</td><td>2</td><td>1500 + 800 + 2200 = 4500</td></tr>
<tr><td>March</td><td>2</td><td>4</td><td>0</td><td>700 + 1500 +800 + 3800 = 6800</td></tr>
<tr><td>(6)</td><td></td><td>5</td><td>1</td><td>700 + 1500 + 800 + 3700 = 6700*</td></tr>
<tr><td></td><td>1</td><td>5</td><td>0</td><td>350 + 1500 + 800 + 3800 = 6450*</td></tr>
<tr><td>Feb</td><td>2</td><td>1</td><td>1</td><td>700 + 1500 + 6450 = 8650*</td></tr>
<tr><td>(2)</td><td></td><td>2</td><td>2</td><td>700 + 1500 + 6700 = 8900</td></tr>
<tr><td></td><td>1</td><td>2</td><td>1</td><td>350 + 1500 + 6450 = 8300*</td></tr>
<tr><td></td><td></td><td>3</td><td>2</td><td>350 + 1500 + 6700 = 8550</td></tr>
<tr><td></td><td>0</td><td>3</td><td>1</td><td>1500 + 6450 = 7950*</td></tr>
<tr><td></td><td></td><td>4</td><td>2</td><td>1500 + 800 + 6700 = 9000</td></tr>
<tr><td>Jan</td><td>0</td><td>3</td><td>0</td><td>1500 + 7950 = 9450*</td></tr>
<tr><td>(3)</td><td></td><td>4</td><td>1</td><td>1500 + 800 + 8300 = 10600</td></tr>
<tr><td></td><td></td><td>5</td><td>2</td><td>1500 + 800 + 8650 = 10950</td></tr>
</table>
</td><td>M1
A1
M1
A1
M1
A1ft
M1
A1ft
M1
A1</td><td>3.1b
1.1b
3.1b
1.1b
1.1b
1.1b
1.1b
1.1b
1.1b
1.1b</td></tr>
<tr><td></td><td>
<table>
<tr><th>Month</th><th>January</th><th>February</th><th>March</th><th>April</th><th>May</th></tr>
<tr><td>Number made</td><td>3</td><td>3</td><td>5</td><td>3</td><td>4</td></tr>
</table>
</td><td>B1</td><td>1.1b</td></tr>
<tr><td></td><td>Minimum production cost: £9450</td><td>B1</td><td>1.1b</td></tr>
<tr><td colspan="4">(12 marks)</td></tr>
</table>

Question 7 notes:

All M marks – must bring optimal result from previous stage into calculations so for the second stage (April) if none of their 2200, 1850 or 2300 (the optimal results from May) are used then M0. Ignore extra rows. Condone and credit rows that have been crossed out if they can still be read. Must have right 'ingredients' (storage costs, additional space costs, overhead cost) at least once per stage. Must have values in two of the three colums (State, Action, Dest). If no working seen then the number stated in the Value column must be correct to imply the correct method has been used

1M1: First stage (May) completed. At least 3 rows, 'something' in each cell (but see M mark guidance above) including the correct structure (e.g. no value greater than 5 in the action column) in each of the first four columns

1A1: CAO for first stage.

2M1: Second stage (April) completed. At least 9 rows, something in each cell (see M mark guidance above) including the correct structure for the fifth (Value) column (e.g. bringing forward values from the previous stage)

2A1: CAO for second stage. No extra rows

3M1: Third stage (March) completed. At least 3 rows, something in each cell (see M mark guidance above)

3A1ft: CAO on the ft for third stage. No extra rows

4M1: Fourth stage (February) completed. At least 6 rows, something in each cell (see M mark guidance above)

4A1ft: CAO on the ft for fourth stage. No extra rows

5M1: Fifth stage (January) completed. At least 3 rows, something in each cell (see M mark guidance above)

5A1: CAO for the fifth stage. No extra rows

1B1: CAO – **but must have scored all previous M marks**

2B1: CAO – condone lack of units **- but must have scored all previous M marks**

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