IYGB - FP1 PAPER J - QUESTION 1

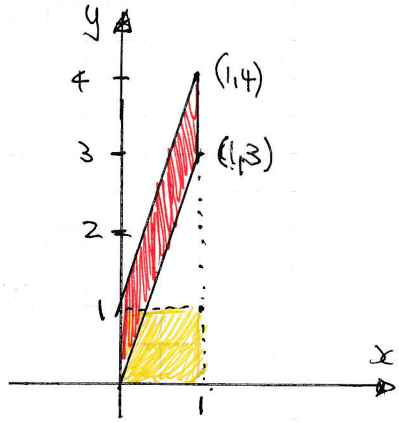
- $\underline{A} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ $\qquad |\underline{A}| = 1$ IE AREA IS PRESERVED, NO REFLECTION

$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

THE MATRIX REPRESENTS A SHEAR PARALLEL TO THE y AXIS, WHERE $(1,0) \longmapsto (1,3)$

$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$



INVARIANT ARE THE POINTS WHICH LIE ON THE y AXIS, IE $x = 0$

- $\underline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45 & -\sin 45 \\ 0 & \sin 45 & \cos 45 \end{bmatrix}$ $\qquad |\underline{B}| = \cos^2 45 + \sin^2 45 = 1$ (NO REFLECTION)

$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x & y & z \end{array}$

$\underline{i} \longmapsto \underline{i}$ & "THE GREEN SECTION" IS A STANDARD ROTATION BY 45° ANTICLOCKWISE ABOUT O, OF THE yz PLANE

THE MATRIX REPRESENTS ROTATION BY 45° ANTICLOCKWISE, ABOUT THE x AXIS, SO THE LINE OF INVARIANT POINTS IS THE x AXIS, WITH EQUATION $y = z = 0$

IYGB - FP1 PAPER J - QUESTION 2

USING THE LINEARITY PROPERTY OF THE SIGMA OPERATOR

$$\sum_{r=1}^{n}(3r^2+r-1) = \sum_{r=1}^{n}3r^2 + \sum_{r=1}^{n}r - \sum_{r=1}^{n}1$$

$$= 3\sum_{r=1}^{n}r^2 + \sum_{r=1}^{n}r - \sum_{r=1}^{n}1$$

$$= 3\times\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - n$$

$$= \frac{1}{2}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - n$$

$$= \frac{1}{2}n\left[(n+1)(2n+1) + (n+1) - 2\right]$$

$$= \frac{1}{2}n\left[2n^2+3n+1+n+1-2\right]$$

$$= \frac{1}{2}n\left[2n^2+4n\right]$$

$$= n(n^2+2n)$$

$$= n^2(n+2)$$

AS REQUIRED

IYGB - FP1 PAPER I - QUESTION 3

a) LOOKING AT THE CUBIC

$$\alpha+\beta+\gamma=-2$$
$$\alpha\beta+\beta\gamma+\gamma\alpha=0$$
$$\alpha\beta\gamma=-k$$

I) $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+\cancel{(\alpha\beta+\beta\gamma+\gamma\alpha)}$

$\therefore \quad \alpha^2+\beta^2+\gamma^2=(-2)^2$

$\alpha^2+\beta^2+\gamma^2 = 4$

AS REQUIRED

II) AS α,β,γ ARE ROOTS

$$\left.\begin{array}{l}\alpha^3+2\alpha^2+k=0\\ \beta^3+2\beta^2+k=0\\ \gamma^3+2\gamma^2+k=0\end{array}\right\}$$

ADDING $\alpha^3+\beta^3+\gamma^3+2(\alpha^2+\beta^2+\gamma^2)+3k=0$

$\alpha^3+\beta^3+\gamma^3+8+3k=0$

$\therefore \quad \alpha^3+\beta^3+\gamma^3=-8-3k$

AS REQUIRED

b) AS $\alpha\beta\gamma=-k\neq0$ THEN $\alpha\neq0,\ \beta\neq0,\ \gamma\neq0$

$$\left.\begin{array}{l}\alpha^3+2\alpha^2+k=0\\ \beta^3+2\beta^2+k=0\\ \gamma^3+2\gamma^2+k=0\end{array}\right\}$$

MULTIPLY THROUGH EACH EQUATION BY α,β & γ RESPECTIVELY

$$\left.\begin{array}{l}\alpha^4+2\alpha^3+\alpha k=0\\ \beta^4+2\beta^3+\beta k=0\\ \gamma^4+2\gamma^3+\gamma k=0\end{array}\right\}$$

ADDING AS BEFORE

$\Longrightarrow (\alpha^4+\beta^4+\gamma^4)+2(\alpha^3+\beta^3+\gamma^3)+(\alpha+\beta+\gamma)k=0$

$\Longrightarrow \quad 8 + 2(-8-3k) + (-2)k=0$

IYGB - FP1 PAPER J - QUESTION 3

$\Rightarrow \quad 8 - 16 - 6k - 2k = 0$

$\Rightarrow \quad -8 = -8k$

$\therefore k = -1$

AS REQUIRED

c) USING THE APPROACH OF PART (b)

$\alpha^5 + 2\alpha^4 + k\alpha^2 = 0$

$\beta^5 + 2\beta^4 + k\beta^2 = 0$

$\gamma^5 + 2\gamma^4 + k\gamma^2 = 0$

ADDING & NOTING $k = -1$

$(\alpha^5 + \beta^5 + \gamma^5) + 2(\alpha^4 + \beta^4 + \gamma^4) + k(\alpha^2 + \beta^2 + \gamma^2) = 0$

$\alpha^5 + \beta^5 + \gamma^5 + 2 \times 4 + (-1) \times 4 = 0$

$\therefore \alpha^5 + \beta^5 + \gamma^5 = -4$

IYGB - FP1 PAPER J - QUESTION 4

USING THE STANDARD FORMULA FOR VOLUME OF REVOLUTION ABOUT THE x AXIS

$$V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx$$

$$V = \pi \int_0^{\ln 2} (4\sqrt{x}e^x)^2 \, dx = \pi \int_0^{\ln 2} 16xe^{2x} \, dx$$

PROCEED BY INTEGRATION BY PARTS

$8x$	8
e^{2x}	$2e^{2x}$

$$V = \pi \int_0^{\ln 2} (8x)(2e^{2x}) \, dx$$

$$V = \pi \left[\left(8xe^{2x}\right)_0^{\ln 2} - \int_0^{\ln 2} 8e^{2x} \, dx \right]$$

$$V = \pi \left[8xe^{2x} - 4e^{2x} \right]_0^{\ln 2}$$

$$V = \pi \left[\left(8\ln 2 \, e^{\ln 4} - 4e^{\ln 4}\right) - (0 - 4) \right]$$

$$V = \pi [32\ln 2 - 16 + 4]$$

$$V = \pi [32\ln 2 - 12]$$

OR $V = 4\pi [-3 + 8\ln 2]$

IYGB - FP1 PAPER J - QUESTION 5

REWRITE THE QUADRATIC

$$\Rightarrow z^2-6z+10+(z-6)i=0$$

$$\Rightarrow z^2-6z+10+iz-6i=0$$

$$\Rightarrow z^2+(i-6)z+(10-6i)=0$$

BY THE QUADRATIC FORMULA

$$\Rightarrow z=\frac{-(i-6)\pm\sqrt{(i-6)^2-4\times1\times(10-6i)}}{2\times1}$$

$$\Rightarrow z=\frac{6-i\pm\sqrt{-1-12i+36-40+24i}}{2}$$

$$\Rightarrow z=\frac{6-i\pm\sqrt{-5+12i}}{2}$$

NOW NEED TO SIMPLIFY THE SQUARE ROOT

$$(a+bi)^2\equiv-5+12i \qquad a,b\in\mathbb{R}$$

$$a^2+2abi-b^2\equiv-5+12i$$

$$\left.\begin{array}{l} a^2-b^2=-5 \\ ab=6 \end{array}\right\} \Rightarrow b=\frac{6}{a}$$

$$\Rightarrow a^2-\left(\frac{6}{a}\right)^2=-5$$

$$\Rightarrow a^2-\frac{36}{a^2}=-5$$

$$\Rightarrow a^4-36=-5a^2$$

$$\Rightarrow a^4+5a^2-36=0$$

IYGB

$\Longrightarrow (a^2 - 4)(a^2 + 9) = 0$

$\Longrightarrow a^2 = \begin{cases} 4 \\ \cancel{-9} \end{cases}$

$\Longrightarrow a = \begin{cases} 2 \\ -2 \end{cases} \qquad b = \frac{6}{a} = \begin{cases} 3 \\ -3 \end{cases}$

FINALLY WE HAVE

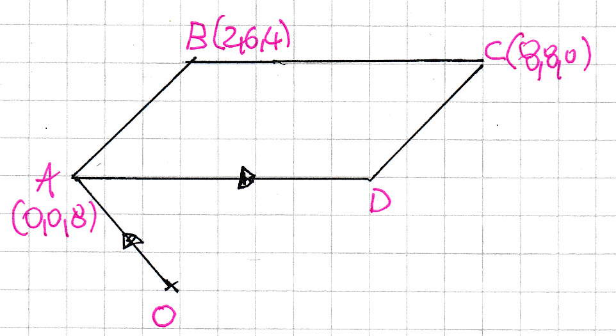
$$z = \frac{6 - i \pm (2+3i)}{2}$$

$$z = \begin{cases} \dfrac{6-i+2+3i}{2} = \dfrac{8+2i}{2} = 4+i \\[2ex] \dfrac{6-i-2-3i}{2} = \dfrac{4-4i}{2} = 2-2i \end{cases}$$

$\therefore \quad z_1 = 4+i \quad \& \quad z_2 = 2-2i$

IYGB – FP1 PAPER J – QUESTION 6

a) LOOKING AT THE DIAGRAM



EITHER BY INSPECTION

$D(6,2,4)$

OR

$\vec{OD} = \vec{OA} + \vec{AD}$

$\vec{OD} = \vec{OA} + \vec{BC}$

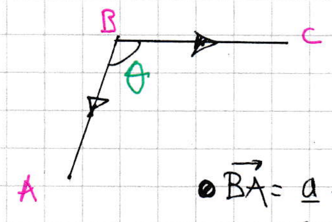
$\vec{OD} = \underline{a} + \underline{c} - \underline{b}$

$\vec{OD} = (0,0,8) + (8,8,0) - (2,6,4)$

$\underline{d} = (6,2,4)$

AS ABOVE

b) LOOKING AT THE DIAGRAM



• $\vec{BC} = \underline{c} - \underline{b}$
$= (8,8,0) - (2,6,4)$
$= (6,2,-4)$

• $\vec{BA} = \underline{a} - \underline{b}$
$= (0,0,8) - (2,6,4)$
$= (-2,-6,4)$

• $(6,2,-4) \cdot (-2,-6,4) = |6,2,-4|\,|-2,-6,4| \cos\theta$

$-12 - 12 - 16 = \sqrt{36+4+16}\,\sqrt{4+36+16}\,\cos\theta$

$-40 = 56 \cos\theta$

$\therefore \cos\theta = -\frac{5}{7}$

• $\sin\theta = +\sqrt{1 - \cos^2\theta}$

$\sin\theta = \sqrt{1 - \frac{25}{49}}$

$\sin\theta = \sqrt{\frac{24}{49}}$

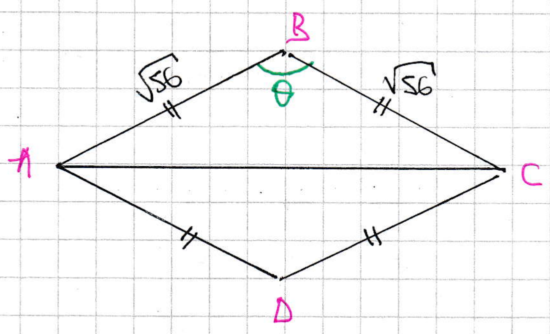
$\sin\theta = \frac{2\sqrt{6}}{7}$

AS REQUIRED

c) BOTH $|AB|$ & $|BC|$ ARE $\sqrt{56}$ SO THE PARALLELOGRAM IS IN FACT A RHOMBUS, SO ITS DIAGONALS MUST BE PERPENDICULAR

IYGB – FP1 PAPER J – QUESTION 6

CONSIDERING THE "RHOMBUS" AS TWO TRIANGLES



$$\text{AREA} = \frac{1}{2} \times |AB||BC| \times \sin\theta \times 2$$

$$\text{AREA} = |AB||BC|\sin\theta$$

$$\text{AREA} = \sqrt{56}\sqrt{56} \times \frac{2}{7}\sqrt{6}$$

$$\text{AREA} = 56 \times \frac{2}{7}\sqrt{6}$$

$$\text{AREA} = 16\sqrt{6}$$

IYGB - FP1 PAPER J - QUESTION 7

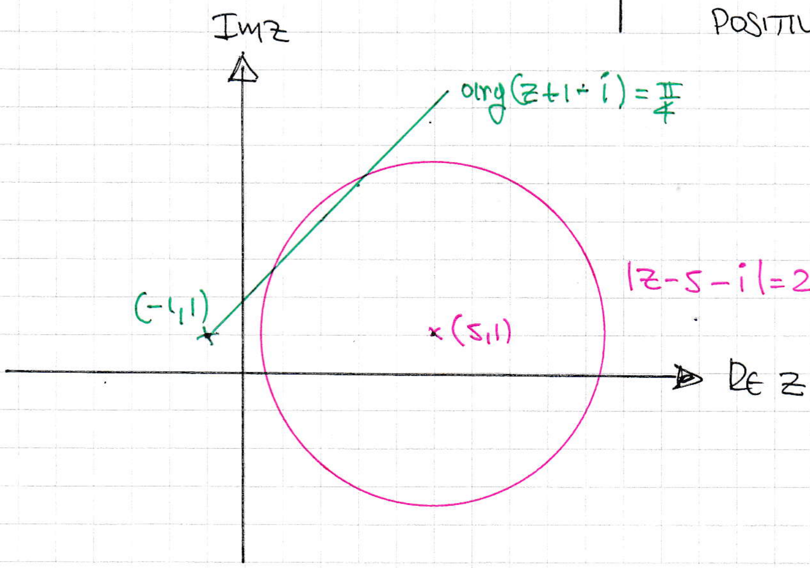
IDENTIFY THE LOCI FIRST

$$|z-5-i| = 2\sqrt{5}$$
$$|z-(5+i)| = \sqrt{20}$$

CIRCLE CENTRE AT $(5,1)$, RADIUS $\sqrt{20}$

$$\arg(z+1-i) = \frac{\pi}{4}$$
$$\arg(z-(-1+i)) = \frac{\pi}{4}$$

HALF LINE, STARTING AT $(-1,1)$ INCLINED AT $\frac{\pi}{4}$ TO THE POSITIVE x AXIS



WORKING IN CARTESIAN

- $y-y_0 = m(x-x_0)$
  $y-1 = 1(x+1)$
  $y-1 = x+1$
  $y = x+2$

- $(x-5)^2+(y-1)^2 = 20$
  $(x-5)^2+(x+2-1)^2 = 20$
  $(x-5)^2+(x+1)^2 = 20$
  $x^2-10x+25+x^2+2x+1 = 20$
  $2x^2-8x+6 = 0$
  $x^2-4x+3 = 0$
  $(x-3)(x-1) = 0$
  $x = \begin{cases}1\\3\end{cases} \quad y = \begin{cases}3\\5\end{cases}$

$\therefore$ $z_1 = 1+3i$ & $z_2 = 3+5i$

IYGB – FP1 PAPER J – QUESTION 8

START WITH THE BASE CASE, $n=1$

$$L.H.S = \sum_{r=1}^{1}\left(\frac{r}{2^r}\right) = \frac{1}{2^1} = \frac{1}{2} \qquad R.H.S. = 2-\frac{n+2}{2^n} = 2-\frac{1+2}{2^1} = \frac{1}{2}$$

$\therefore$ THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n=k \in \mathbb{N}$

$$\Longrightarrow \sum_{r=1}^{k}\left(\frac{r}{2^r}\right) = 2-\frac{k+2}{2^k}$$

$$\Longrightarrow \left[\sum_{r=1}^{k}\left(\frac{r}{2^r}\right)\right] + \frac{k+1}{2^{k+1}} = 2-\frac{k+2}{2^k}+\frac{k+1}{2^{k+1}}$$

$$\Longrightarrow \sum_{r=1}^{k+1}\left(\frac{r}{2^r}\right) = 2+\left[\frac{k+1}{2^{k+1}}-\frac{k+2}{2^k}\right] = 2+\left[\frac{(k+1)-2(k+2)}{2^{k+1}}\right]$$

$$\Longrightarrow \sum_{r=1}^{k+1}\left(\frac{r}{2^r}\right) = 2+\frac{-k-3}{2^{k+1}} = 2-\frac{k+3}{2^{k+1}}$$

$$\Longrightarrow \sum_{r=1}^{k+1}\left(\frac{r}{2^r}\right) = 2-\frac{(k+1)+2}{2^{k+1}}$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$

SINCE THE RESULT HOLDS ~~HOLDS~~ FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

IYGB - FP1 PAPER J - QUESTION 9

WORKING AS FOLLOWS

- "OBJECT LINE" $y = mx$
- "IMAGE LINE" $Y = mX$

$$\Longrightarrow \begin{pmatrix} -2 & 1 \\ -9 & 4 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$$

$$\Longrightarrow \begin{aligned} -2x + mx &= X \\ -9x + 4mx &= mX \end{aligned}$$

DIVIDING THE EQUATIONS

$$\Longrightarrow \frac{-2+m}{-9+4m} = \frac{1}{m}$$

$$\Longrightarrow -2m + m^2 = 4m - 9$$

$$\Longrightarrow m^2 - 6m + 9 = 0$$

$$\Longrightarrow (m-3)^2 = 0$$

$$\Longrightarrow m = 3$$

$\therefore$ REQUIRED LINE

$$y = 3x$$