IYGB - FP2 PAPER Q - QUESTION 1

START BY FINDING AN INTEGRATING FACTOR

$$\text{I.F.} = e^{\int \frac{4}{x}\,dx} = e^{4\ln x} = e^{\ln x^4} = x^4$$

MULTIPLY THROUGH WE OBTAIN

$$x^4 \frac{dy}{dx} + x^4\left(\frac{4y}{x}\right) = x^4(6x-5)$$

$$x^4 \frac{dy}{dx} + 4x^3 y = 6x^5 - 5x^4$$

$$\frac{d}{dx}\left(x^4 y\right) = 6x^5 - 5x^4$$

$$x^4 y = \int 6x^5 - 5x^4 \, dx$$

$$x^4 y = x^6 - x^5 + C$$

$$y = x^2 - x + \frac{C}{x^4}$$

APPLY CONDITION $x=1, y=1$

$$1 = 1 - 1 + \frac{C}{1}$$

$$\therefore C = 1$$

THUS WE HAVE

$$y = x^2 - x + \frac{1}{x^4}$$

IYGB – FP2 PAPER Q – QUESTION 2

PROCEED AS FOLLOWS

$$\int_0^e x^2 \ln x \, dx = \lim_{h \to 0^+} \left[\int_h^e x^2 \ln x \, dx \right]$$

INTEGRATION BY PARTS

$\ln x$	$\frac{1}{x}$
$\frac{1}{3}x^3$	x^2

$$\dots = \lim_{h \to 0^+} \left[\left[\frac{1}{3}x^3 \ln|x| \right]_h^e - \int_h^e \frac{1}{3}x^2 \, dx \right]$$

$$= \lim_{h \to 0^+} \left[\left[\frac{1}{3}x^3 \ln|x| - \frac{1}{9}x^3 \right]_h^e \right]$$

$$= \lim_{h \to 0^+} \left[\left(\frac{1}{3}e^3 \ln e - \frac{1}{9}e^3 \right) - \left(\frac{1}{3}h^3 \ln h - \frac{1}{9}h^3 \right) \right]$$

$$= \lim_{h \to 0^+} \left[\frac{1}{3}e^3 - \frac{1}{9}e^3 - \frac{1}{3}h^3 \ln h + \frac{1}{9}h^3 \right]$$

NOW LOOKING AT THE TERM $h^3 \ln h$, h^3 TENDS TO ZERO "FASTER" THAN $\ln h$ TENDS TO $-\infty$, I.E $\lim_{h \to 0^+} \left(h^3 \ln h \right) = 0$

$$\dots = \frac{1}{3}e^3 - \frac{1}{9}e^3 - 0 + 0$$

$$= \frac{2}{9}e^3$$

AS REQUIRED

IYGB - FP2 PAPER Q - QUESTION 3

START BY "SPLITTING THE FRACTION"

$$f(x) = \frac{e^x + 1}{2e^{\frac{1}{2}x}} = \frac{e^x}{2e^{\frac{1}{2}x}} + \frac{1}{2e^{\frac{1}{2}x}} = \frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{2}e^{-\frac{1}{2}x}$$

$$= \cosh\left(\tfrac{1}{2}x\right)$$

NOW $\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + O(u^8)$

$$f(x) = 1 + \frac{\left(\frac{1}{2}x\right)^2}{2!} + \frac{\left(\frac{1}{2}x\right)^4}{4!} + \frac{\left(\frac{1}{2}x\right)^6}{6!} + O(x^8)$$

$$f(x) = 1 + \frac{1}{8}x^2 + \frac{1}{384}x^4 + \frac{1}{46080}x^6 + O(x^8)$$

ALTERNATIVE USING EXPONENTIALS

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + O(x^4)$
- $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + O(x^4)$

$$\therefore f(x) = \tfrac{1}{2}e^{\frac{1}{2}x} + \tfrac{1}{2}e^{-\frac{1}{2}x} = \tfrac{1}{2}\left(e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}\right)$$

$$= \frac{1}{2}\left[\begin{matrix} 1 + \frac{1}{2}x + \frac{\frac{1}{4}x^2}{2!} + \frac{\frac{1}{8}x^3}{3!} + \frac{\frac{1}{16}x^4}{4!} + \frac{\frac{1}{32}x^5}{5!} + \frac{\frac{1}{64}x^6}{6!} + O(x^7) \\ 1 - \frac{1}{2}x + \frac{\frac{1}{4}x^2}{2!} - \frac{\frac{1}{8}x^3}{3!} + \frac{\frac{1}{16}x^4}{4!} - \frac{\frac{1}{32}x^5}{5!} + \frac{\frac{1}{64}x^6}{6!} + O(x^7) \end{matrix}\right]$$

$$= \frac{1}{2}\left[2 + \frac{1}{4}x^2 + \frac{1}{192}x^4 + \frac{1}{23040}x^6 + O(x^8)\right]$$

$$= 1 + \frac{1}{8}x^2 + \frac{1}{384}x^4 + \frac{1}{46080}x^6 + O(x^8)$$

AS ABOVE

IYGB - FP2 PAPER Q - QUESTION 4

PROCEED BY IDENTITIES

$\Longrightarrow 2\tanh^2 w = 1 + \operatorname{sech} w$

$\Longrightarrow 2(1 - \operatorname{sech}^2 w) = 1 + \operatorname{sech} w$

$\Longrightarrow 2 - 2\operatorname{sech}^2 w = 1 + \operatorname{sech} w$

$\Longrightarrow 0 = 2\operatorname{sech}^2 w + \operatorname{sech} w - 1$

$\Longrightarrow (2\operatorname{sech} w - 1)(\operatorname{sech} w + 1)$

$\Longrightarrow \operatorname{sech} w = \begin{cases} -1 \\ \frac{1}{2} \end{cases}$

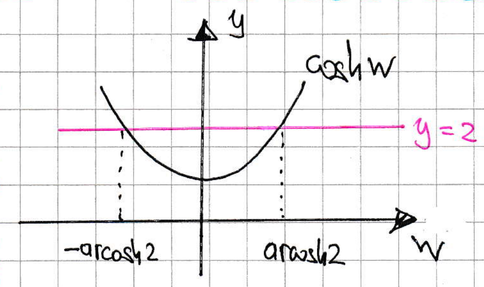
$\Longrightarrow \cosh w = \begin{cases} \cancel{-1} \\ 2 \end{cases} \quad \longleftarrow (\cosh w \geq 1)$

$\Longrightarrow w = \pm \operatorname{arcosh} 2$

$\Longrightarrow w = \pm \ln\left(2 + \sqrt{2^2 - 1}\right)$

$\Longrightarrow w = \pm \ln\left(2 + \sqrt{3}\right)$

$1 + \tan^2\theta \equiv \sec^2\theta$
$1 - \tanh^2\theta \equiv \operatorname{sech}^2\theta$
$1 - \operatorname{sech}^2\theta \equiv \tanh^2\theta$



IYGB - FP2 PAPER Q - QUESTION 5

$$\boxed{\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 10\sin x \qquad y(0)=6, \ y'(0)=5}$$

AUXILIARY EQUATION

$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda - 2)(\lambda - 1) = 0$$
$$\lambda = \begin{cases} 1 \\ 2 \end{cases}$$

COMPLEMENTARY FUNCTION

$$y = Ae^{x} + Be^{2x}$$

PARTICULAR INTEGRAL BY INSPECTION

$$y = P\cos x + Q\sin x$$
$$y' = -P\sin x + Q\cos x$$
$$y'' = -P\cos x - Q\sin x$$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow (-P\cos x - Q\sin x) - 3(-P\sin x + Q\cos x) + 2(P\cos x + Q\sin x) \equiv 10\sin x$$

$$\Rightarrow \left\{\begin{array}{l} -P\cos x - Q\sin x \\ -3Q\cos x + 3P\sin x \\ +2P\cos x + 2Q\sin x \end{array}\right\} \equiv 10\sin x$$

$$\Rightarrow (P - 3Q)\cos x + (3P + Q)\sin x \equiv 10\sin x$$

- $P - 3Q = 0$
  $P = 3Q$

- $3P + Q = 10$
  $3(3Q) + Q = 10$
  $10Q = 10$
  $\underline{Q = 1}$ & $\underline{P = 3}$

IYGB - FP2 PAPER Q - QUESTION 5

∴ PARTICULAR INTEGRAL

$$y = 3\cos x + \sin x$$

∴ GENERAL SOLUTION

$$y = Ae^{x} + Be^{2x} + 3\cos x + \sin x$$

DIFFERENTIATE W.R.T x & APPLY CONDITIONS

$$\frac{dy}{dx} = Ae^{x} + 2Be^{2x} - 3\sin x + \cos x$$

- $x=0,\ y=6 \implies 6 = A + B + 3$
  $\implies A + B = 3$

- $x=0,\ \frac{dy}{dx}=5 \implies 5 = A + 2B + 1$
  $\implies A + 2B = 4$

$$\therefore B = 1 \qquad A = 2$$

FINALLY WE OBTAIN

$$y = 2e^{x} + e^{2x} + 3\cos x + \sin x$$

IYGB - FP2 PAPER Q - QUESTION 6

$$\boxed{\text{USING} \quad \frac{d}{dx}\left(\arctan f(x)\right) = \frac{1}{1+[f(x)]^2} \times f'(x)}$$

$$\frac{d}{dx}\left[\arctan\left(\frac{x}{\sqrt{4-x^2}}\right)\right] = \frac{d}{dx}\left[\arctan\left[\frac{x}{(4-x^2)^{\frac{1}{2}}}\right]\right]$$

$$= \frac{1}{1+\frac{x^2}{4-x^2}} \times \frac{(4-x^2)^{\frac{1}{2}} \times 1 - x \times \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)}{(4-x^2)^1}$$

$$= \frac{1}{1+\frac{x^2}{4-x^2}} \times \frac{(4-x^2)^{\frac{1}{2}} + x^2(4-x^2)^{-\frac{1}{2}}}{(4-x^2)}$$

$$= \frac{(4-x^2)^{\frac{1}{2}} + x^2(4-x^2)^{-\frac{1}{2}}}{(4-x^2)+x^2}$$

$$= \frac{(4-x^2)^{-\frac{1}{2}}\left[\cancel{(4-x^2)^1 + x^2}\right]}{\cancel{(4-x^2)+x^2}}$$

$$= \frac{1}{\sqrt{4-x^2}}$$

IYGB - FP2 PAPER Q - QUESTION 7

a) USING THE STANDARD FORMULA FOR THE AREA IN POLARS

$$\text{AREA} = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}(\sin 2\theta)^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin^2 2\theta \, d\theta$$

NOW USING THE TRIGONOMETRIC IDENTITY FOR COSINE DOUBLE ANGLE

$$\cos 2A \equiv 1 - 2\sin^2 A$$

$$\cos 4A \equiv \cos[2(2A)] \equiv 1 - 2\sin^2 2A$$

$$\sin^2 2A \equiv \frac{1}{2} - \frac{1}{2}\cos 4A$$

$$\text{AREA} = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2}\cos 4\theta\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4} - \frac{1}{4}\cos 4\theta$$

$$= \left[\frac{1}{4}\theta - \frac{1}{16}\sin 4\theta\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} \times \frac{1}{4} - 0\right) - (0 - 0)$$

$$= \frac{\pi}{8}$$

IYGB - FP2 PAPER Q - QUESTION 7

b) FOR "HORIZONTAL TANGENT" $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = 0$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = \frac{d}{d\theta}(\sin 2\theta \sin\theta) = 0$$

DIFFERENTIATE & SOLVE THE EQUATION

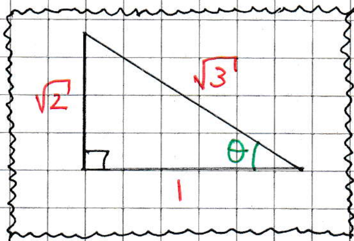
$$\Rightarrow 2\cos 2\theta \sin\theta + \sin 2\theta \cos\theta = 0$$

$$\Rightarrow 2\sin\theta(2\cos^2\theta - 1) + 2\sin\theta\cos^2\theta = 0$$

$$\Rightarrow 2\sin\theta\left[3\cos^2\theta - 1\right] = 0$$

$\therefore$ $\cancel{\sin\theta = 0}$ $\qquad \cos\theta = \frac{1}{\sqrt{3}}$ $\qquad \cancel{\cos\theta = -\frac{1}{\sqrt{3}}}$

$\therefore$ $\theta = \arccos\left(\frac{1}{\sqrt{3}}\right)$



$\therefore$ $r = \sin 2\theta = 2\sin\theta\cos\theta$

$$= 2 \times \frac{\sqrt{2}}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = \frac{2}{3}\sqrt{2}$$

POLAR CO-ORDINATES OF $P\left(\frac{2}{3}\sqrt{2}, \arccos\frac{1}{\sqrt{3}}\right)$

$\uparrow r \qquad \uparrow \theta$

$$x = r\cos\theta = \frac{2}{3}\sqrt{2} \times \frac{1}{\sqrt{3}} = \frac{2\sqrt{2}}{3\sqrt{3}} = \frac{2}{9}\sqrt{6}$$

$$y = r\sin\theta = \frac{2}{3}\sqrt{2}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{4}{3\sqrt{3}} = \frac{4}{9}\sqrt{3}$$

CARTESIAN CO-ORDINATES $\left(\frac{2}{9}\sqrt{6}, \frac{4}{9}\sqrt{3}\right)$

IYGB – FP2 PAPER Q – QUESTION 8

USING THE HINT GIVEN

$$(a-1)^3 = a^3 - 3a^2 \times 1 + 3a \times 1^2 - 1^3$$
$$(a-1)^3 = a^3 - 3a^2 + 3a - 1$$

COMPARING WE OBTAIN

$$\Rightarrow z^3 - 3z^2 + 3z - 65 = 0$$
$$\Rightarrow \underbrace{z^3 - 3z^2 + 3z - 1}_{} - 64 = 0$$
$$(z-1)^3 = 64$$

USING EXPONENTIALS

$$\Rightarrow (z-1)^3 = 64e^{0i + 2k\pi i} \qquad k = 0,1,2$$
$$\Rightarrow (z-1)^3 = 64e^{2k\pi i}$$
$$\Rightarrow z - 1 = 64^{\frac{1}{3}} e^{\frac{2k\pi}{3}i}$$
$$\Rightarrow z = 1 + 4e^{\frac{2}{3}k\pi i}$$

- $z_0 = 1 + 4e^0 = 1 + 4 = 5$
- $z_1 = 1 + 4e^{\frac{2}{3}\pi i} = 1 + 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 1 + 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$$= -1 + 2\sqrt{3}i$$
- $z_2 = 1 + 4e^{\frac{4}{3}\pi i} = 1 + 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 1 + 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$
$$= -1 - 2\sqrt{3}i$$

$$\therefore z = \begin{cases} 5 \\ -1 + 2\sqrt{3}i \\ -1 - 2\sqrt{3}i \end{cases}$$

IYGB - FP2 PAPER Q - QUESTION 9

a) $$f(r) = \frac{1}{(r+1)(r-1)} = \frac{\frac{1}{2}}{r-1} - \frac{\frac{1}{2}}{r+1} = \frac{1}{2(r-1)} - \frac{1}{2(r+1)}$$

"BY COVER UP" (INSPECTION)

b) USING PART (a) "DOUBLED" FOR SIMPLICITY

$$\boxed{\frac{2}{r^2-1} \equiv \frac{2}{(r+1)(r-1)} \equiv \frac{1}{r-1} - \frac{1}{r+1}}$$

- $r=2$ $\quad \frac{2}{2^2-1} = \frac{1}{1} - \cancel{\frac{1}{3}}$
- $r=3$ $\quad \frac{2}{3^2-1} = \frac{1}{2} - \cancel{\frac{1}{4}}$
- $r=4$ $\quad \frac{2}{4^2-1} = \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$
- $r=5$ $\quad \frac{2}{5^2-1} = \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$
- $r=6$ $\quad \frac{2}{6^2-1} = \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$

$\vdots$

- $r=n-1$ $\quad \frac{2}{(n-1)^2-1} = \cancel{\frac{1}{n-2}} - \frac{1}{n}$
- $r=n$ $\quad \frac{2}{n^2-1} = \cancel{\frac{1}{n-1}} - \frac{1}{n+1}$

$$\sum_{r=2}^{n} \frac{2}{r^2-1} = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$2\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$$

c) AS $n \to \infty$ THE SUM TEND TO $\frac{3}{4}$