IYGB - FP3 PAPER L - QUESTION 1

METHOD A

$$\frac{(2x-1)(x+3)}{(x-3)(x-2)} < 2 \implies \frac{(2x-1)(x+3)}{(x-3)(x-2)} - 2 < 0$$

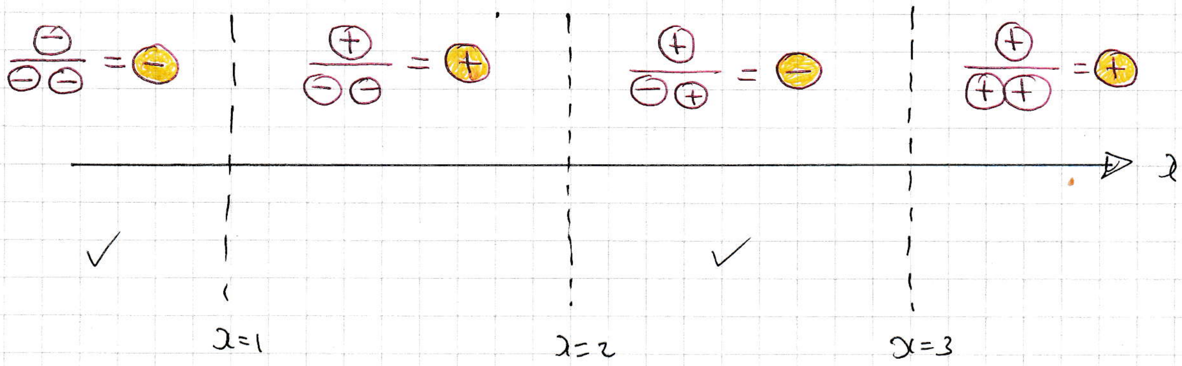
$$\implies \frac{(2x-1)(x+3)-2(x-3)(x-2)}{(x-3)(x-2)} < 0$$

$$\implies \frac{\cancel{2x^2}+5x-3-\cancel{2x^2}+10x-12}{(x-3)(x-2)} < 0$$

$$\implies \frac{15x-15}{(x-3)(x-2)} < 0$$

$$\implies \frac{x-1}{(x-3)(x-2)} < 0$$

THE CRITICAL VALUES ARE 1, 2, 3



HENCE WE HAVE

$$x < 1 \quad \cup \quad 2 < x < 3$$

IYGB - FP3 PAPER L - QUESTION 1

METHOD B

$$\Rightarrow \frac{(2x-1)(x+3)}{(x-3)(x-2)} < 2$$

$$\Rightarrow \frac{(2x-1)(x+3)(x-3)(x-2)}{(x-3)^2(x-2)^2} < 2$$

WE MAY MULTIPLY ACROSS AS THE DENOMINATOR IS NOW POSITIVE

$$\Rightarrow (2x-1)(x+3)(x-3)(x-2) < 2(x-3)^2(x-2)^2$$

$$\Rightarrow (2x-1)(x+3)(x-3)(x-2) - 2(x-3)^2(x-2)^2 < 0$$

$$\Rightarrow (x-3)(x-2)\left[(2x-1)(x+3) - 2(x-3)(x-2)\right] < 0$$

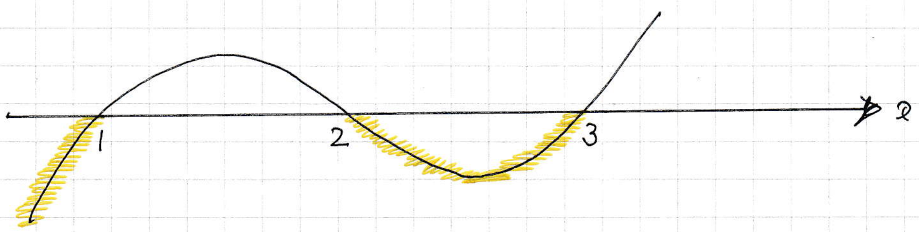
$$\Rightarrow (x-3)(x-2)\left[2x^2+5x-3 - 2(x^2-5x+6)\right] < 0$$

$$\Rightarrow (x-3)(x-2)\left[\cancel{2x^2}+5x-3-\cancel{2x^2}+10x-12\right] < 0$$

$$\Rightarrow (x-3)(x-2)(15x-15) < 0$$

$$\Rightarrow 15(x-1)(x-2)(x-3) < 0$$

SKETCHING THE CUBIC ABOVE



$$\therefore \quad x < 1 \quad \text{OR} \quad 2 < x < 3$$

IYGB - FP3 PAPER L - QUESTION 2

$$\boxed{\frac{dy}{dx} = e^{x} - y^{2} \qquad x=0,\ y=0\ ,\ h=0.1}$$

a) USING THE RESULT $y_{n+1} = hy_n' + y_n$

$$\Rightarrow y_1 = hy_0' + y_0 \qquad (x_0=0,\ y_0=0)$$
$$\Rightarrow y_1 = 0.1(e^{x_0} - y_0^2) + y_0$$
$$\Rightarrow y_1 = 0.1(e^0 - 0^2) + 0$$
$$\Rightarrow y_1 = 0.1$$

IE $y \simeq 0.1$ AT $x=0.1$

b) NEXT USING THE RESULT $y_n' = \frac{y_{n+1} - y_{n-1}}{2h}$

$$\Rightarrow \boxed{y_{n+1} = 2hy_n' + y_{n-1}}$$
$$\Rightarrow y_2 = 2hy_1' + y_0$$
$$\Rightarrow y_2 = 2 \times 0.1 \times (e^{x_1} - y_1^2) + y_0 \qquad (x_1=0.1,\ y_1=0.1)$$
$$\Rightarrow y_2 = 0.2(e^{0.1} - 0.1^2) + 0$$
$$\Rightarrow y_2 = 0.219034\ldots$$

$$\Rightarrow y_3 = 2hy_2' + y_1 \qquad (x=0.2,\ y_2=0.2190\ldots)$$
$$\Rightarrow y_3 = 2 \times 0.1 \times (e^{x_2} - y_2^2) + y_1$$
$$\Rightarrow y_3 = 0.2 \times (e^{0.2} - 0.219\ldots^2) + 0.1$$
$$\Rightarrow y_3 = 0.3346853\ldots$$

$\therefore$ THE APPROXIMATE VALUE OF y AT $x=0.3$ IS 0.3347

IYGB - FP3 PAPER L - QUESTION 2

b) FINDING THE FIRST 4 DERIVATIVES

	$x_0 = 0 \quad y_0 = 0$
$y' = e^x - y^2$	$y_0' = e^{x_0} - y_0$ $y_0' = e^0 - 0$ $y_0' = 1$
$y'' = e^x - 2yy'$	$y_0'' = e^{x_0} - 2y_0 y_0'$ $y_0'' = e^0 - 0$ $y_0'' = 1$
$y''' = e^x - 2y'y' - 2yy''$	$y_0''' = e^{x_0} - 2y_0'y_0' - 2y_0y_0''$ $y_0''' = e^0 - 2\times1\times1 - 0$ $y_0''' = -1$
$y'''' = e^x - 2(y')^2 - 2yy''$ $y'''' = e^x - 4y'y'' - 2y'y'' - 2yy'''$ $y'''' = e^x - 6y'y'' - 2yy'''$	$y_0'''' = e^{x_0} - 6y_0'y_0'' - 2y_0y_0'''$ $y_0'''' = e^0 - 6\times1\times1 - 0$ $y_0'''' = -5$

$$\Rightarrow y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0'''' + O(x^4)$$

$$\Rightarrow y = x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{5}{24}x^4 + O(x^4)$$

$$\Rightarrow y(0.3) \approx 0.3 + \frac{1}{2}(0.3)^2 - \frac{1}{6}(0.3)^3 - \frac{5}{24}(0.3)^4 \simeq 0.3388$$

IYGB - FP3 PAPER L - QUESTION 3

USING STANDARD EXPANSION

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4)$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + O(x^4)$$

$$\sin x = x - \frac{1}{6}x^3 + O(x^5) \; ; \; \cos x = 1 - \frac{1}{2}x^2 + O(x^4)$$

$$\sin^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\left[1 - \frac{1}{2}(2x)^2 + O(x^4)\right]$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\left[1 - 2x^2 + O(x^4)\right]$$

$$\sin^2 x = \cancel{\frac{1}{2} - \frac{1}{2}} + x^2 + O(x^4)$$

THUS WE NOW HAVE

$$\lim_{x \to 0}\left[\frac{\ln(1-x)}{\sin^2 x} + \operatorname{cosec} x\right] = \lim_{x \to 0}\left[\frac{\ln(1-x) + \operatorname{cosec} x \sin^2 x}{\sin^2 x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\ln(1-x) + \sin x}{\sin^2 x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\left[-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + O(x^4)\right] + \left[x - \frac{1}{6}x^3 + O(x^5)\right]}{x^2 + O(x^4)}\right]$$

$$= \lim_{x \to 0}\left[\frac{-\frac{1}{2}x^2 - \frac{5}{6}x^3 + O(x^4)}{x^2 + O(x^4)}\right]$$

$$= \lim_{x \to 0}\left[\frac{-\frac{1}{2} - \frac{5}{6}x + O(x^2)}{1 + O(x^2)}\right]$$

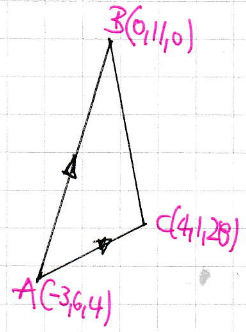
$$= -\frac{1}{2}$$

IYGB - FP3 PAPER L - QUESTION 4

a) START BY WORKING OUT THE RELEVANT VECTORS

$$\vec{AB} = \underline{b} - \underline{a} = (0,11,0) - (-3,6,4) = (3,5,-4)$$

$$\vec{AC} = \underline{c} - \underline{a} = (4,1,28) - (-3,6,4) = (7,-5,24)$$



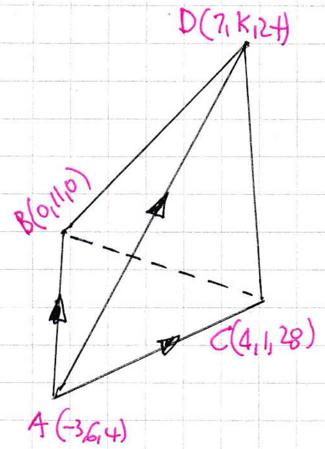
HENCE USING THE STANDARD FORMULA

$$\text{AREA} = \frac{1}{2}\left|\vec{AB} \wedge \vec{AC}\right| = \frac{1}{2}\left\|\begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 5 & -4 \\ 7 & -5 & 24 \end{matrix}\right\| = \frac{1}{2}\left|120-20, -28-72, -15-35\right|$$

$$= \frac{1}{2}\left|100,-100,-50\right| = \frac{1}{2}\times 50\left|2,-2,1\right| = 25\sqrt{4+4+1} = \underline{75}$$

b) WORK OUT THE VECTOR $\vec{AD}$ IN TERMS OF k

$$\vec{AD} = \underline{d} - \underline{a} = (7,k,24) - (-3,6,4) = (10, k-6, 20)$$



USING THE "STANDARD FORMULA" FOR TETRAHEDRON

$$\text{VOLUME} = \frac{1}{6}\left|\vec{AB} \wedge \vec{AC} \cdot \vec{AD}\right|$$

$$= \frac{1}{6}\left\|\begin{matrix} 10 & k-6 & 20 \\ 3 & 5 & -4 \\ 7 & -5 & 24 \end{matrix}\right\|$$

$$= \frac{1}{6}\left|(100,-100,-50)\cdot(10,k-6,20)\right|$$

$$= \frac{1}{6}\left|\cancel{1000} - 100(k-6) - \cancel{1000}\right|$$

$$= \frac{1}{6}\left|-100\right|\left|k-6\right|$$

$$= \underline{\frac{50}{3}\left|k-6\right|}$$

IYGB – FP3 PAPER L – QUESTION 4

c) USING PART (b)

$$\frac{50}{3}|k-6| = 150$$

$$50|k-6| = 450$$

$$|k-6| = 9$$

$$k-6 = \begin{cases} 9 \\ -9 \end{cases}$$

$$k = \begin{cases} 15 \\ -3 \end{cases}$$

IYGB - FP3 PAPER L - QUESTION 5

a) DETERMINE THE GRADIENT FUNCTION PARAMETRICALLY

$$\left.\begin{array}{l} x = a\sec\theta \\ y = b\tan\theta \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$$

$$= (b\sec\theta)\left(\tfrac{1}{a}\cot\theta\right) = \frac{b}{\cos\theta} \times \frac{1}{a}\frac{\cos\theta}{\sin\theta}$$

$$= \frac{b}{a\sin\theta}$$

HENCE THE GRADIENT AT THE GENERAL POINT $(a\sec\theta, b\tan\theta)$ IS $\frac{b}{a\sin\theta}$

NORMAL $\Rightarrow y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$

$$\Rightarrow by - b^2\tan\theta = -ax\sin\theta + a^2\sin\theta\sec\theta$$

$$\Rightarrow by + ax\sin\theta = b^2\tan\theta + a^2\sin\theta \times \frac{1}{\cos\theta}$$

$$\Rightarrow by + ax\sin\theta = b^2\tan\theta + a^2\tan\theta$$

$$\Rightarrow by + ax\sin\theta = (a^2+b^2)\tan\theta$$

AS REQUIRED

b) FIND THE CO-ORDS OF A & B

$x=0 \Rightarrow by = (a^2+b^2)\tan\theta$

$$\Rightarrow y = \frac{a^2+b^2}{b}\tan\theta$$

$A\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$

$y=0 \Rightarrow ax\sin\theta = (a^2+b^2)\tan\theta$

$$ax\sin\theta = (a^2+b^2)\frac{\sin\theta}{\cos\theta}$$

$$ax = \frac{a^2+b^2}{\cos\theta}$$

$$x = \frac{a^2+b^2}{a}\sec\theta$$

$B\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$

IYGB - FP3 PAPER L - QUESTION 5

THE MIDPOINT OF AB IS $M\left(\frac{a^2+b^2}{2a}\sec\theta,\ \frac{a^2+b^2}{2b}\tan\theta\right)$

$\uparrow x \qquad \uparrow y$

$$x = \frac{a^2+b^2}{2a}\sec\theta$$

$$\frac{2ax}{a^2+b^2} = \sec\theta$$

$$y = \frac{a^2+b^2}{2b}\tan\theta$$

$$\frac{2by}{a^2+b^2} = \tan\theta$$

$\Rightarrow 1 + \tan^2\theta = \sec^2\theta$

$\Rightarrow 1 + \left(\frac{2by}{a^2+b^2}\right)^2 = \left(\frac{2ax}{a^2+b^2}\right)^2$

$\Rightarrow 1 + \frac{4b^2y^2}{(a^2+b^2)^2} = \frac{4a^2x^2}{(a^2+b^2)^2}$

$\Rightarrow (a^2+b^2)^2 + 4b^2y^2 = 4a^2x^2$

$\Rightarrow (a^2+b^2)^2 = 4a^2x^2 - 4b^2y^2$

$\Rightarrow 4(a^2x^2 - b^2y^2) = (a^2+b^2)^2$

AS REQUIRED

IYGB - FP3 PAPER L - QUESTION 6

a) USING THE SUBSTITUTION GIVEN

$$z = \frac{1}{y^2} \implies \frac{d}{dx}(z) = \frac{d}{dx}\left(\frac{1}{y^2}\right)$$

$$\implies \frac{dz}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$$

$$\implies \frac{dy}{dx} = -\frac{y^3}{2}\frac{dz}{dx}$$

SUBSTITUTE INTO THE O.D.E.

$$\frac{1}{y}\frac{dy}{dx} = 1 + 2xy^2$$

$$\frac{dy}{dx} = y + 2xy^3$$

$$-\frac{y^3}{2}\frac{dz}{dx} = y + 2xy^3 \qquad \times\left(-\frac{2}{y^3}\right)$$

$$\frac{dz}{dx} = -\frac{2}{y^2} - 4x$$

$$\frac{dz}{dx} = -2z - 4x$$

$$\frac{dz}{dx} + 2z = -4x$$

// AS REQUIRED

b) LOOKING FOR AN INTEGRATING FACTOR

$$\text{I.F.} = e^{\int 2\,dx} = e^{2x}$$

$$\implies \frac{d}{dx}\left(ze^{2x}\right) = -4xe^{2x}$$

$$\implies ze^{2x} = \int -4xe^{2x}\,dx$$

IYGB - FP3 PAPER L - QUESTION 6

INTEGRATION BY PARTS ON THE R.H.S.

$$\int -4xe^{2x}\,dx = -2xe^{2x} - \int -2e^{2x}\,dx$$

$$= -2xe^{2x} + \int 2e^{2x}\,dx$$

$$= e^{2x} - 2xe^{2x} + C$$

$-4x$	-4
$\frac{1}{2}e^{2x}$	e^{2x}

RETURNING TO THE O.D.E.

$$ze^{2x} = e^{2x} - 2xe^{2x} + C$$

$$z = 1 - 2x + Ce^{-2x}$$

$$\frac{1}{y^2} = 1 - 2x + Ce^{-2x}$$

$$y^2 = \frac{1}{1 - 2x + Ce^{-2x}}$$

IYGB - FP3 PAPER L - QUESTION 7

a) WORKING AS FOLLOWS

$$\text{LHS} = \sec x = \frac{1}{\cos x} = \frac{1}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} = \frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \frac{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} + \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}} - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{1 + \tan^2\frac{x}{2}}{1 - \tan^2\frac{x}{2}} = \text{R.H.S.}$$

[OR LITERALLY WORK IN REVERSE FROM THE R.H.S TO L.H.S]

b) BY INSPECTION / COVER UP OR ANY SENSIBLE METHOD

$$\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)} = \frac{1}{1+t} + \frac{1}{1-t}$$

c) USING THE SUBSTITUTION GIVEN

- $t = \tan\frac{x}{2} \implies \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$$\implies \frac{dt}{dx} = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)$$

$$\implies \frac{dt}{dx} = \frac{1}{2}(1+t^2)$$

$$\implies \frac{dx}{dt} = \frac{2}{1+t^2}$$

$$\implies \boxed{dx = \frac{2}{1+t^2}\,dt}$$

IYGB - FP3 PAPER L - QUESTION 7

USING PARTS (a) & (b)

$$\int \sec x \, dx = \int \frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}} \, dx = \int \frac{\cancel{1+t^2}}{1-t^2} \times \frac{2}{\cancel{1+t^2}} \, dt$$

$$= \int \frac{2}{1-t^2} \, dt = \int \frac{1}{1+t} + \frac{1}{1-t} \, dt$$

$$= \ln|1+t| - \ln|1-t| + C$$

$$= \ln\left|\frac{1+t}{1-t}\right| + C$$

NOW NOTING THAT $\tan\frac{x}{2} = t$ & $\tan\frac{\pi}{4} = 1$

$$\ldots = \ln\left|\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}\right| + C$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

// AS REQUIRED