IYGB - FP3 PAPER N - QUESTION 1

USING THE FACT THAT THE "CROSS PRODUCT" IS DISTRIBUTIVE OVER ADDITION & SUBTRACTION, WE OBTAIN

$$(2\underline{a}+\underline{b})\wedge(\underline{a}-2\underline{b}) = 2\underline{a}\wedge\underline{a} - 4\underline{a}\wedge\underline{b} + \underline{b}\wedge\underline{a} - 2\underline{b}\wedge\underline{b}$$

NEXT WE USE THE PROPERTIES

- $\underline{u}\wedge\underline{u} = \underline{0}$ FOR ALL $\underline{u}$
- $\underline{u}\wedge\underline{v} = -\underline{v}\wedge\underline{u}$ FOR ALL $\underline{u}$ & $\underline{v}$

$$\dots = \underline{0} + 4\underline{b}\wedge\underline{a} + \underline{b}\wedge\underline{a} - \underline{0}$$

$$= 5\underline{b}\wedge\underline{a}$$

[OR INDEED $-5\underline{a}\wedge\underline{b}$]

IYGB - FP3 PAPER M - QUESTION 2

$$\boxed{\text{AREA} = \int_0^1 \frac{1}{1+\sqrt{x}}\,dx}$$

FORM A TABLE OF VALUES FOR THE INTEGRAND

x	0	0.25	0.5	0.75	1
y	1	$\frac{2}{3}$	$2-\sqrt{2}$	$4-2\sqrt{3}$	$\frac{1}{2}$
	FIRST	ODD	EVEN	ODD	LAST

USING SIMPSON'S FORMULA

$$\Rightarrow \text{AREA} \approx \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODDS} + 2\times\text{EVENS}\right]$$

$$\Rightarrow \text{AREA} \approx \frac{0.25}{3}\left[1+\frac{1}{2}+4\left(\frac{2}{3}+4-2\sqrt{3}\right)+2\times\left(2-\sqrt{2}\right)\right]$$

$$\Rightarrow \text{AREA} \approx 0.623486\ldots$$

$$\Rightarrow \text{AREA} \approx 0.623$$

3. d.p.

IYGB - FP3 PAPER M - QUESTION 3

SOLVING IN THE USUAL WAY

$$\frac{x+k}{x+4k} > \frac{k}{x} \implies \frac{x+k}{x+4k} - \frac{k}{x} > 0$$

$$\implies \frac{x(x+k) - k(x+4k)}{x(x+4k)} > 0$$

$$\implies \frac{x^2+kx-kx-4k^2}{x(x+4k)} > 0$$

$$\implies \frac{x^2-4k^2}{x(x+4k)} > 0$$

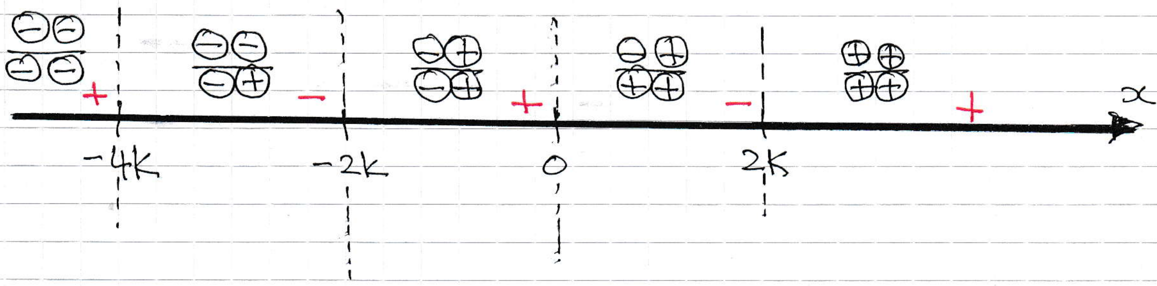
$$\implies \frac{(x-2k)(x+2k)}{x(x+4k)} > 0$$

THE CRITICAL VALUES OF x FOR THIS INEQUALITY ARE

$x = 2k, \ -2k$ (x INTERCEPTS FROM THE NUMERATOR)

$x = 0, \ -4k$ (VERTICAL ASYMPTOTES FROM THE DENOMINATOR)

USING A NUMBER LINE TO CHECK THE INTERVALS



$x < -4k$ OR $-2k < x < 0$ OR $x > 2k$

IYGB - FP3 PAPER M - QUESTION 4

a) START BY OBTAINING DERIVATIVES & THEIR EVALUATIONS AT $x=1$, AS THE EXPANSION IS IN POWERS OF $(x-1)$

- $f(x) = x^2 \ln x$
  $f(1) = 1^2 \times \cancel{\ln 1} = 0$
- $f'(x) = 2x\ln x + x^2\left(\frac{1}{x}\right) = 2x\ln x + x$
  $f'(1) = 2\times1\times\cancel{\ln 1} + 1 = 1$
- $f''(x) = 2\ln x + 2x\left(\frac{1}{x}\right) + 1 = 2\ln x + 2 + 1 = 2\ln x + 3$
  $f''(1) = \cancel{2\ln 1} + 3 = 3$
- $f'''(x) = \frac{2}{x}$
  $f'''(1) = 2$

HENCE WE CAN OBTAIN AN EXPANSION

$$\Rightarrow f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \dots$$

$$\Rightarrow x^2\ln x = 0 + (x-1)\times 1 + \frac{(x-1)^2}{2}\times 3 + \frac{(x-1)^3}{6}\times 2 + \dots$$

$$\Rightarrow x^2\ln x = (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots$$

b) LET $x = 1.1$ IN THE ABOVE EXPANSION GIVES

$$\Rightarrow (1.1)^2 \ln(1.1) \simeq (0.1) + \frac{3}{2}(0.1)^2 + \frac{1}{3}(0.1)^3$$

$$\Rightarrow 1.21 \ln(1.1) \simeq \frac{173}{1500}$$

$$\Rightarrow \ln(1.1) \approx \frac{173}{1815} \approx 0.095$$

IYGB - FP3 PAPER M - QUESTION 5

- WRITE THE EQUATIONS IN PARAMETRIC

$$\mathbf{r}_1 = (2,-1,1) + \lambda(0,1,3) = (2,\lambda-1,3\lambda+1)$$
$$\mathbf{r}_2 = (1,2,3) + \mu(1,0,2) = (\mu+1,2,2\mu+3)$$

- EQUATE $\underline{i}$

$\mu+1=2$
$\mu=1$

- EQUATE $\underline{j}$

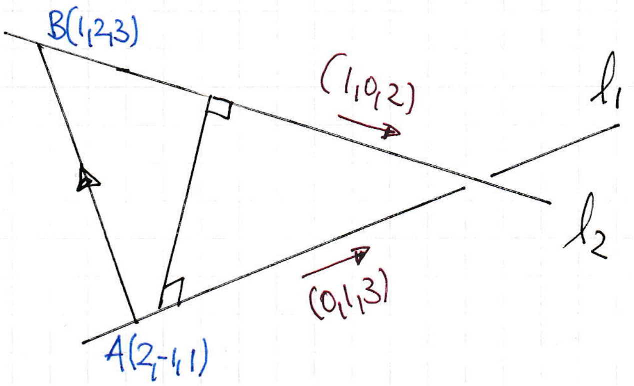
$\lambda-1=2$
$\lambda=3$

- CHECK $\underline{k}$, WITH $\mu=1, \lambda=3$

$3\lambda+1=10$
$2\mu+3=5$

$\therefore$ LINES ARE SKEW

- DRAWING A DIAGRAM OF THE TWO SKEW LINES - FIND THE COMMON PERPENDICULAR BY CROSSING THEIR DIRECTION VECTORS



$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = (-2,-3,1)$$

- NORMALIZING & SCALING GIVES

$$\frac{1}{\sqrt{2^2+3^2+1^2}}(2,3,-1) = \frac{1}{\sqrt{14}}(2,3,-1)$$

- HENCE WE HAVE BY PROJECTING $\overrightarrow{AB}$ ONTO A UNIT NORMAL PERPENDICULAR

$$\overrightarrow{AB} = \underline{b} - \underline{a} = (1,2,3)-(2,-1,1) = (-1,3,2)$$

$$d_{MIN} = \left|(-1,3,2)\cdot\frac{1}{\sqrt{14}}(2,3,-1)\right| = \frac{1}{\sqrt{14}}\left|-2+9-2\right| = \frac{5}{\sqrt{14}}$$

IYGB - FP3 PAPER M - QUESTION 6

$y = x^3 e^{2x}$ TO BE DIFFERENTIATED k TIMES

PICK SENSIBLE CHOICES FOR u & v

$u = e^{2x}$ (DIFFERENTIATING ANY NUMBER OF TIMES IS EASY)

$v = x^3$ (AFTER 4 DIFFERENTIATIONS IT VANISHES

LEIBNIZ RULE STATES

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{d^1 v}{dx^1} + \frac{n(n-1)}{2!}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2 v}{dx^2} + \dots$$

IN THIS QUESTION

$$\frac{d^k}{dx^k}(x^3e^{2x}) = 2^k e^{2x} \cdot x^3 + k \cdot 2^{k-1}e^{2x}(3x^2) + \frac{k(k-1)}{2!}2^{k-2}e^{2x}(6x)$$

$$+ \frac{k(k-1)(k-2)}{3!}2^{k-3}e^{2x}(6) + \text{"REST IS ZERO"}$$

$$= e^{2x}\left[2^k x^3 + 3kx^2 2^{k-1} + \frac{1}{2}k(k-1) \times 6x2^{k-2} + \frac{1}{6}k(k-1)(k-2) \times 6 \times 2^{k-3}\right]$$

$$= e^{2x}\left[x^3 2^k + 3kx^2 2^{k-1} + 3k(k-1)x2^{k-2} + k(k-1)(k-2)2^{k-3}\right]$$

$$= e^{2x}2^{k-3}\left[x^3 2^3 + 3kx^2 2^2 + 3k(k-1)x \times 2^1 + k(k-1)(k-2)\right]$$

$$= e^{2x}2^{k-3}\left[8x^3 + 12kx^2 + 6k(k-1)x + k(k-1)(k-2)\right]$$

IYGB – FP3 PAPER M – QUESTION 7

a) START BY OBTAIN "REPLACEMENTS" FOR $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$

- $x = \ln t$

DIFFERENTIATE W.R.T y

$$\Rightarrow \frac{dx}{dy} = \frac{1}{t}\frac{dt}{dy}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = t\frac{dy}{dt}}$$

- ALSO NOTE THAT

$$x = \ln t$$

$$\frac{dx}{dt} = \frac{1}{t}$$

$$\boxed{\frac{dt}{dx} = t}$$

- $\frac{dy}{dx} = t\frac{dy}{dt}$

DIFFERENTIATE W.R.T x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dt}{dx}\frac{dy}{dt} + t\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + t\frac{d^2y}{dt^2}\times\frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = t\frac{dy}{dt} + t\frac{d^2y}{dt^2}\times t$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} = t\frac{dy}{dt} + \frac{d^2y}{dt^2}t^2}$$

SUBSTITUTING INTO THE O.D.E. AND SIMPLIFY, NOTING FURTHER THAT $e^x = t$

$$\Rightarrow \frac{d^2y}{dx^2} - (1-6e^x)\frac{dy}{dx} + 10ye^{2x} = 5e^{2x}\sin(2e^x)$$

$$\Rightarrow \left[t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}\right] - (1-6t)\left(t\frac{dy}{dt}\right) + 10yt^2 = 5t^2\sin(2t)$$

$$\Rightarrow t^2\frac{d^2y}{dt^2} + \cancel{t\frac{dy}{dt}} - \cancel{t\frac{dy}{dt}} + 6t^2\frac{dy}{dt} + 10t^2y = 5t^2\sin 2t$$

$$\Rightarrow \cancel{t^2}\frac{d^2y}{dt^2} + 6\cancel{t^2}\frac{dy}{dt} + 10\cancel{t^2}y = 5\cancel{t^2}\sin 2t$$

$$\Rightarrow \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 5\sin 2t$$

AS REQUIRED

IYGB - FP3 PAPER M - QUESTION 7

b) SOLVING THE TRANSFORMED EQUATION

- AUXILIARY EQUATION

$\Rightarrow \lambda^2 + 6\lambda + 10 = 0$

$\Rightarrow (\lambda+3)^2 - 9 + 10 = 0$

$\Rightarrow (\lambda+3)^2 = -1$

$\Rightarrow \lambda + 3 = \pm i$

$\Rightarrow \lambda = -3 \pm i$

COMPLEMENTARY FUNCTION

$y = e^{-3t}(A\cos t + B\sin t)$

- PARTICULAR INTEGRAL

$y = P\cos 2t + Q\sin 2t$

$\dot{y} = -2P\sin 2t + 2Q\cos 2t$

$\ddot{y} = -4P\cos 2t - 4Q\sin 2t$

SUB INTO THE O.D.E.

$\ddot{y} = -4P\cos 2t - 4Q\sin 2t$

$+6\dot{y} = 12Q\cos 2t - 12P\sin 2t$

$+10y = 10P\cos 2t + 10Q\sin 2t$

ADDING AND COMPARING

$$(6P+12Q)\cos 2t + (6Q-12P)\sin 2t \equiv 5\sin 2t$$

$$\left.\begin{array}{l} 6P+12Q=0 \\ 6Q-12P=5 \end{array}\right\} \Rightarrow P = -2Q$$

$\Rightarrow 6Q + 24Q = 5$

$\Rightarrow 30Q = 5$

$\Rightarrow Q = \frac{1}{6}$

$\Rightarrow P = -\frac{1}{3}$

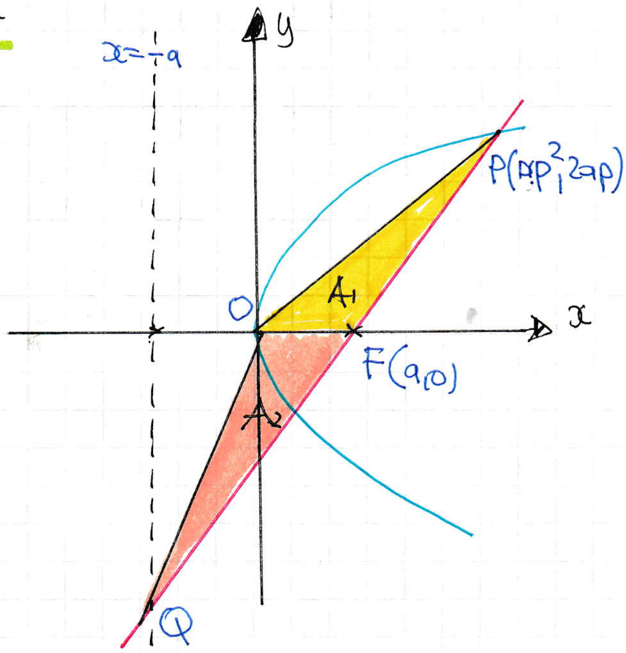
HENCE THE GENERAL SOLUTION CAN BE FOUND

$\Rightarrow y = e^{-3t}(A\cos t + B\sin t) - \frac{1}{3}\cos 2t + \frac{1}{6}\sin 2t$

$\Rightarrow y = e^{-3e^x}\left[A\cos(e^x) + B\sin(e^x)\right] - \frac{1}{3}\cos(2e^x) + \frac{1}{6}\sin(2e^x)$

IYGB - FP3 PAPER N - QUESTION 8

- START WITH A DIAGRAM AND USING THE STANDARD PARAMETRIZATION FOR A PARABOLA, WE OBTAIN



$$\text{GRAD PF} = \frac{2ap-0}{ap^2-a} = \frac{2p}{p^2-1}$$

EQUATION OF PF

$$y - 0 = \frac{2p}{p^2-1}(x-a)$$

WHEN $x=-a$

$$y = \frac{2p}{p^2-1}(-a-a)$$

$$y = \frac{-4ap}{p^2-1}$$

$$\therefore Q\left(-a, \frac{-4ap}{p^2-1}\right)$$

- COMPUTE THE INDIVIDUAL AREAS A_1 & A_2 (DIAGRAM)

$$\left.\begin{aligned} A_1 &= \tfrac{1}{2} \times a \times 2ap = a^2p \\ A_2 &= \tfrac{1}{2} \times a \times \frac{4ap}{p^2-1} = \frac{2a^2p}{p^2-1} \end{aligned}\right\} \Rightarrow A_1 + A_2 = \frac{15}{4}a^2$$

$$\Rightarrow a^2p + \frac{2a^2p}{p^2-1} = \frac{15}{4}a^2$$

$$\Rightarrow p + \frac{2p}{p^2-1} = \frac{15}{4}$$

$$\Rightarrow 4p + \frac{8p}{p^2-1} = 15$$

IYGB - FP3 PAPER N - QUESTION 8

$$\Longrightarrow \quad 4p(p^2-1) + 8p = 15(p^2-1)$$

$$\Longrightarrow \quad 4p^3 - 4p + 8p = 15p^2 - 15$$

$$\Longrightarrow \quad 4p^3 - 15p^2 + 4p + 15 = 0$$

- THIS IS DIFFICULT TO FACTORIZE SO WE USE THE "SLOW"

$$\begin{array}{r|l} & 4p^2 - 3p - 5 \\ \hline p-3 & 4p^3 - 15p^2 + 4p + 15 \\ & -4p^3 + 12p^2 \\ \hline & \quad -3p^2 + 4p + 15 \\ & \quad 3p^2 - 9p \\ \hline & \qquad -5p + 15 \\ & \qquad 5p - 15 \\ \hline & \qquad\qquad 0 \end{array}$$

$$\Longrightarrow \quad (4p^2-3p-5)(p-3) = 0$$

- BY THE QUADRATIC FORMULA

$$p = \frac{3 \pm \sqrt{9-4\times4(-5)}}{2\times4} = \frac{3 \pm \sqrt{89}}{8} \quad //$$

IYGB - FP3 PAPER M - QUESTION 9

$$\boxed{\frac{d^2x}{dt^2} = -x \quad , \quad \text{SUBJECT TO THE CONDITIONS} \quad \begin{array}{l} t_0 = 0 \\ x_0 = 0 \\ \dot{x}_0 = x'_0 = 1 \end{array}}$$

USE THE TWO FORMULAS

$$\boxed{\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}} \quad \& \quad \boxed{\frac{d^2y}{dx^2} \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}}$$

HERE WITH OUR VARIABLES, SINCE $x = x(t)$

$$\boxed{x'_0 \approx \frac{x_1 - x_{-1}}{2h}} \quad \& \quad \boxed{x''_0 \approx \frac{x_1 - 2x_0 + x_{-1}}{h^2}}$$

FROM THE O.D.E ITSELF $x''_0 = -x_0 = 0$

HENCE WE OBTAIN FROM EACH FORMULA, WITH $h = 0.1$

$$1 = \frac{x_1 - x_{-1}}{2} \quad \& \quad 0 = \frac{x_1 - 0 + x_{-1}}{(0.1)^2}$$

$$x_1 - x_{-1} = 0.2 \qquad\qquad x_1 + x_{-1} = 0$$

$$2x_1 = 0.2$$

$$x_1 = 0.1 \quad \& \quad x_{-1} = -0.1$$

HENCE THE SECOND DERIVATIVE APPROXIMATION FORMULA YIELDS

$$\Longrightarrow x''_0 = \frac{x_1 - 2x_0 + x_{-1}}{h^2}$$

$$\Longrightarrow -x_0 = \frac{x_1 - 2x_0 + x_{-1}}{h^2}$$

IYGB - FP3 PAPER M - QUESTION 9

$$\Rightarrow -h^2 x_0 = x_1 - 2x_0 + x_{-1}$$

$$\Rightarrow x_1 = 2x_0 - h^2 x_0 - x_{-1}$$

$$\Rightarrow \boxed{x_1 = x_0(2-h^2) - x_{-1}}$$

OR AS A RECURRENCE RELATION

$$\Rightarrow x_{n+2} = x_{n+1}(2-h^2) - x_n \quad , \text{ WITH } \quad x_0 = 0, \; x_1 = 0.1, \; h = 0.1$$

$$\Rightarrow \boxed{x_{n+2} = 1.99\,x_{n+1} - x_n}$$

APPLY THE RESULT RECURSIVELY

$$\Rightarrow x_2 = 1.99x_1 - x_0$$
$$\Rightarrow x_2 = 0.199 - 0$$
$$\Rightarrow x_2 = 0.199$$

$$\Rightarrow x_3 = 1.99x_2 - x_1$$
$$\Rightarrow x_3 = 1.99(0.199) - 0.1$$
$$\Rightarrow x_3 = 0.29601$$

$$\Rightarrow x_4 = 1.99x_3 - x_2$$
$$\Rightarrow x_4 = 1.99(0.29601) - 0.199$$
$$\Rightarrow x_4 = 0.3900599\ldots$$

$$\Rightarrow x_5 = 1.99x_4 - x_3$$
$$\Rightarrow x_5 = 1.99(0.3900599\ldots) - 0.29601\ldots$$
$$\Rightarrow x_5 = 0.4802090\ldots$$

$\therefore$ AT $t = 0.5 \quad x \simeq 0.480$