IYGB - FP3 PAPER N - QUESTION 1

FORMING A TABLE OF VALUES FOR THE INTEGRAND, USING 7 ORDINATES (6 STRIPS)

x	1	1.25	1.50	1.75	2	2.25	2.5
$\sqrt{x^3+1}$	1.4142	1.7185	2.0917	2.5218	3	3.5200	4.0774
	FIRST	ODD	EVEN	ODD	EVEN	ODD	LAST

USING SIMPSON'S RULE

$$\int_1^{2.5} \sqrt{x^3+1}\,dx \approx \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + 2\times\text{EVEN} + 4\times\text{ODDS}\right]$$

$$\approx \frac{0.25}{3}\left[1.4142 + 4.0774 + 2(2.0917+3) + 4(1.7185+2.5218+3.5200)\right]$$

$$\approx \frac{1}{12}\times 46.7162\ldots$$

$$\simeq 3.8930\ldots$$

$$\simeq 3.89$$

3 s.f.

IYGB - FP3 PAPER N - QUESTION 2

DIFFERENTIATE THE O.D.E IN SUCCESSION AND EVALUATE THE DERIVATIVES AT $x=0$

DIFFERENTIATIONS	EVALUATIONS
	$y_0 = 2$ (GIVEN)
$y' = x^2 - y^2$	$y_0' = x_0^2 - y_0^2$ $y_0' = 0^2 - 2^2$ $y_0' = -4$
$y'' = 2x - 2yy'$	$y_0'' = 2x_0 - 2y_0y_0'$ $y_0'' = 2\times0 - 2\times2\times(-4)$ $y_0'' = 16$
$y''' = 2 - 2y'y' - 2yy''$	$y_0''' = 2 - 2y_0'y_0' - 2y_0y_0''$ $y_0''' = 2 - 2(-4)^2 - 2\times2\times16$ $y_0''' = -94$

EXPANDING AS A POWER SERIES

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + O(x^4)$$

$$y = 2 + x(-4) + \frac{x^2}{2}(16) + \frac{x^3}{6}(-94) + O(x^4)$$

$$y = 2 - 4x + 8x^2 - \frac{47}{3}x^3 + O(x^4)$$

IYGB - FP3 PAPER N - QUESTION 3

AS THE DENOMINATOR IS UNDER THE MODULUS SIGN, I.E IT IS NON-NEGATIVE, WE MAY MULTIPLY ACROSS

$$\Longrightarrow \frac{5x-1}{|2x-3|} \geqslant 1$$

$$\Longrightarrow 5x-1 \geqslant |2x-3|$$

SOLVING THE CORRESPONDING EQUATION TO OBTAIN THE CRITICAL VALUES OF THE INEQUALITY

$$5x-1 = |2x-3|$$

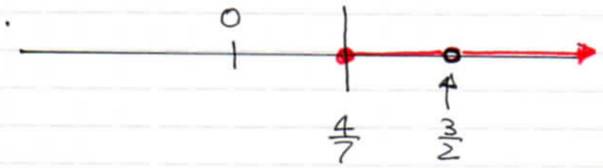
$$\left(\begin{array}{l} 5x-1 = 2x-3 \\ 5x-1 = 3-2x \end{array}\right) \Longrightarrow \left(\begin{array}{l} 3x = -2 \\ 7x = 4 \end{array}\right)$$

$$\Longrightarrow x = \begin{cases} \cancel{-\frac{2}{3}} \\ \frac{4}{7} \end{cases}$$

DOES NOT SATISFY THE ORIGINAL

ONLY CRITICAL VALUE IS $x = \frac{4}{7}$ — CHECK IF SAY 0 WORKS

$$5(0)-1 = -1$$
$$|2\times 0-3| = 3$$



$$\therefore \quad x \geqslant \frac{4}{7}, \quad x \neq \frac{3}{2}$$

IYGB - FP3 PAPER N - QUESTION 4

a) DIFFERENTIATING IMPLICITLY & FIND GRADIENT AT P

$$y^2 = 12x$$

$$2y\frac{dy}{dx} = 12$$

$$\frac{dy}{dx} = \frac{6}{y}$$

$$\left.\frac{dy}{dx}\right|_{y=6t} = \frac{6}{6t} = \frac{1}{t}$$

EQUATION OF THE TANGENT AT THE GENERAL POINT $P(3t^2, 6t)$

$$\Longrightarrow \ y - y_0 = m(x - x_0)$$

$$\Longrightarrow \ y - 6t = \frac{1}{t}(x - 3t^2)$$

$$\Longrightarrow \ yt - 6t^2 = x - 3t^2$$

$$\Longrightarrow \ yt = x + 3t^2$$

AS REQUIRED

b) I) START BY OBTAINING THE COORDINATES OF Q & S

- WHEN $x=0$
  $yt = 0 + 3t^2$
  $y = 3t$
  $\therefore Q(0, 3t)$

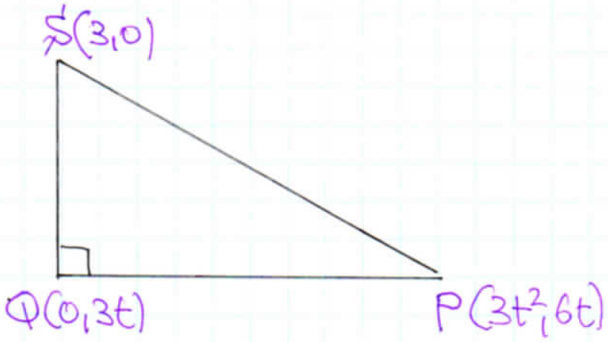
- $y^2 = 12x$
  $y^2 = 4 \times 3 \times x \quad (y^2 = 4ax)$
  $\therefore$ FOCUS $S(3, 0)$

- GRADIENT $PQ = \dfrac{6t - 3t}{3t^2 - 0} = \dfrac{3t}{3t^2} = \dfrac{1}{t}$ (INDEED AS EXPECTED!)
- GRADIENT $SQ = \dfrac{3t - 0}{0 - 3} = \dfrac{3t}{-3} = -t$

AS THEIR GRADIENTS ARE NEGATIVE RECIPROCALS OF ONE ANOTHER PQ IS PERPENDICULAR TO SQ

IYGB - FP3 PAPER N - QUESTION 4

II) DRAWING A DIAGRAM TO RECORD THE INFO



- $|PQ| = \sqrt{(0-3t^2)^2+(3t-6t)^2} = \sqrt{9t^4+9t^2}$
- $|SQ| = \sqrt{(3-0)^2+(0-3t)^2} = \sqrt{9+9t^2}$
- $\text{AREA} = \frac{1}{2}|PQ||SQ| = \frac{1}{2}\sqrt{9t^4+9t^2}\sqrt{9+9t^2}$

$$= \frac{1}{2}|3t|\sqrt{t^2+1} \times 3\sqrt{1+t^2}$$

$$= \frac{9}{2}|t|(t^2+1)$$

AS REQUIRED

$\boxed{\sqrt{x^2} \equiv |x|}$

ALTERNATIVE FOR b II

AREA OF TRIANGLE WITH VERTICES AT $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ IS GIVEN BY

$$\frac{1}{2}\left\|\begin{matrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix}\right\|$$

$$\text{AREA} = \frac{1}{2}\left\|\begin{matrix} 1 & 1 & 1 \\ 3 & 0 & 3t^2 \\ 0 & 3t & 6t \end{matrix}\right\| = \frac{1}{2}\left| 3\times 3t \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & t^2 \\ 0 & 1 & 2 \end{vmatrix}\right|$$

$$= \frac{9}{2}\left| t\left[\begin{vmatrix} 0 & t^2 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right]\right|$$ ← EXPANDING BY THE FIRST COLUMN

$$= \frac{9}{2}\left|t(-t^2-1)\right| = \frac{9}{2}\left|-t(t^2+1)\right| = \frac{9}{2}|t|(t^2+1)$$

IYGB - FP3 PAPER N - QUESTION 5

USING THE SUBSTITUTION GIVEN

$\Rightarrow y = xV(x)$

$\Rightarrow \frac{dy}{dx} = 1 \times V(x) + x\frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = V + x\frac{dv}{dx}$

SUBSTITUTING INTO THE O.D.E.

$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{3x+2xv}{3xv-2x}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{3+2v}{3v-2}$

$\Rightarrow x\frac{dv}{dx} = \frac{3+2v}{3v-2} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{3+2v-3v^2+2v}{3v-2}$

$\Rightarrow x\frac{dv}{dx} = \frac{3+4v-3v^2}{3v-2}$

SEPARATING VARIABLES

$\Rightarrow \int \frac{3v-2}{3+4v-3v^2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \int \frac{3v-2}{3v^2-4v-3}\,dv = \int -\frac{1}{x}\,dx$

$\Rightarrow \int \frac{6v-4}{3v^2-4v-3}\,dv = \int -\frac{2}{x}\,dx$

$\Rightarrow \ln|3v^2-4v-3| = -2\ln|x| + \ln A$

$\Rightarrow \ln|3v^2-4v-3| = \ln\left(\frac{1}{x^2}\right) + \ln A$

$\Rightarrow \ln|3v^2-4v-3| = \ln\left|\frac{A}{x^2}\right|$

$\Rightarrow 3v^2-4v-3 = \frac{A}{x^2}$

REVERSING THE TRANSFORMATION

$\Rightarrow 3\left(\frac{y}{x}\right)^2 - 4\left(\frac{y}{x}\right) - 3 = \frac{A}{x^2}$

$\Rightarrow \frac{3y^2}{x^2} - \frac{4y}{x} - 3 = \frac{A}{x^2}$

$\Rightarrow 3y^2 - 4xy - 3x^2 = A$

APPLY CONDITION $(1,3)$

$3\times3^2 - 4\times1\times3 - 3\times1^2 = A$

$27 - 12 - 3 = A$

$A = 12$

$\therefore 3y^2 - 4xy - 3x^2 = 12$

IYGB - FP3 PAPER N - QUESTION 5

ALTERNATIVE SUBSTITUTION

$$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$$

$$\Rightarrow 3\frac{dy}{dx} = \frac{9x+6y}{3y-2x}$$

$$\Rightarrow \frac{dv}{dx}+2 = \frac{9x+(2v+4x)}{v}$$

$$\Rightarrow \frac{dv}{dx}+2 = \frac{2v+13x}{v}$$

$$\Rightarrow \frac{dv}{dx}+\cancel{2} = \cancel{2}+\frac{13x}{v}$$

$$\Rightarrow \int v \, dv = \int 13x \, dx$$

$$\Rightarrow \frac{1}{2}v^2 = \frac{13}{2}x^2 + C$$

$$\Rightarrow v^2 = 13x^2 + C$$

$$\Rightarrow (3y-2x)^2 = 13x^2 + C$$

APPLY $(1,3)$ $\Rightarrow (9-2)^2 = 13\times1^2+C$

$\Rightarrow 49 = 13 + C$

$\Rightarrow C = 36$

$$\Rightarrow (3y-2x)^2 = 13x^2+36$$

$$\Rightarrow 9y^2-12xy+4x^2 = 13x^2+36$$

$$\Rightarrow 9y^2-12xy-9x^2 = 36$$

$$\Rightarrow 3y^2-4xy-3x^2 = 12$$

AS BEFORE

LET $v(x) = 3y-2x$

$$\frac{dv}{dx} = 3\frac{dy}{dx}-2$$

$$3\frac{dy}{dx} = \frac{dv}{dx}+2$$

ALSO WE HAVE

$$3y = v+2x$$

$$6y = 2v+4x$$

IYGB - FP3 PAPER N - QUESTION 5

ALTERNATIVE BY MULTIVARIABLE CALCULUS

$$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$$

$$\Rightarrow (3y-2x)\,dy = (3x+2y)\,dx$$

$$\Rightarrow (3x+2y)\,dx + (2x-3y)\,dy = 0$$

$$\frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy = dG$$

EVIDENTLY THIS IS EXACT AS $\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x} = 2$

HENCE WE HAVE BY DIRECT INTEGRATION

- $dG = 0 \Rightarrow G(x,y) = \text{constant}$
- $\frac{\partial G}{\partial x} = 3x+2y \Rightarrow G(x,y) = \frac{3}{2}x^2 + 2xy + f(y)$
- $\frac{\partial G}{\partial y} = 2x-3y \Rightarrow G(x,y) = 2xy - \frac{3}{2}y^2 + g(x)$

$$\therefore f(y) = -\frac{3}{2}y^2 \quad \& \quad g(x) = \frac{3}{2}x^2$$

THUS WE OBTAIN

$$G(x,y) = \text{constant}$$

$$\frac{3}{2}x^2 + 2xy - \frac{3}{2}y^2 = \text{constant}$$

$$3x^2 + 4xy - 3y^2 = \text{constant}$$

(1,3) NOW YIELDS
$3 + 12 - 27 = \text{constant}$
$\text{constant} = -12$

$$3x^2 + 4xy - 3y^2 = -12$$

$$3y^2 - 4xy - 3x^2 = 12$$

AS BEFORE

IYGB – FP3 PAPER N – QUESTION 6

a) CALCULATE THE RELEVANT VECTORS FOR A CROSS PRODUCT

$$\vec{AB} = \underline{b} - \underline{a} = (9,1,0) - (5,1,3) = (4,0,-3)$$
$$\vec{AD} = \underline{d} - \underline{a} = (-3,8,6) - (5,1,3) = (-8,7,3)$$

$$\text{AREA} = |\vec{AB} \wedge \vec{AD}| = \left\| \begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 0 & -3 \\ -8 & 7 & 3 \end{matrix} \right\| = |21, 12, 28|$$

$$= \sqrt{21^2 + 12^2 + 28^2} = \sqrt{1369} = 37$$

b) VOLUME IS $|\vec{AE} \cdot (\vec{AB} \wedge \vec{AD})|$, SO WE OBTAIN

$$\Rightarrow V = |\vec{AE} \cdot (21, 12, 28)|$$
$$\Rightarrow V = |(\underline{e} - \underline{a}) \cdot (21, 12, 28)|$$
$$\Rightarrow V = |[(7,2,9) - (5,1,3)] \cdot (21, 12, 28)|$$
$$\Rightarrow V = |(2,1,6) \cdot (21, 12, 28)|$$
$$\Rightarrow V = |42 + 12 + 168|$$
$$\Rightarrow V = 222$$

AS REQUIRED

c) WE SHOULD OBTAIN THE VOLUME AS

$$\Rightarrow V = \text{BASE AREA} \times \text{HEIGHT}$$
$$\Rightarrow 222 = 37 \times h$$
$$\Rightarrow h = 6$$

∴ THE REQUIRED DISTANCE IS 6

IYGB - FP3 PAPER N - QUESTION 7

START BY TRIGONOMETRIC IDENTITIES FIRST

$$\lim_{x \to 0}\left[\frac{\cos^2 3x - 1}{x^2}\right] = \lim_{x \to 0}\left[\frac{\left(\frac{1}{2}+\frac{1}{2}\cos 6x\right) - 1}{x^2}\right]$$

$$= \lim_{x \to 0}\left[\frac{\frac{1}{2}\cos 6x - \frac{1}{2}}{x^2}\right] = \lim_{x \to 0}\left[\frac{\cos 6x - 1}{2x^2}\right]$$

USING THE STANDARD EXPANSION OF $\cos x = 1 - \frac{x^2}{2!} + O(x^4)$

$$= \lim_{x \to 0}\left[\frac{\left[1 - \frac{(6x)^2}{2!} + O(x^4)\right] - 1}{2x^2}\right]$$

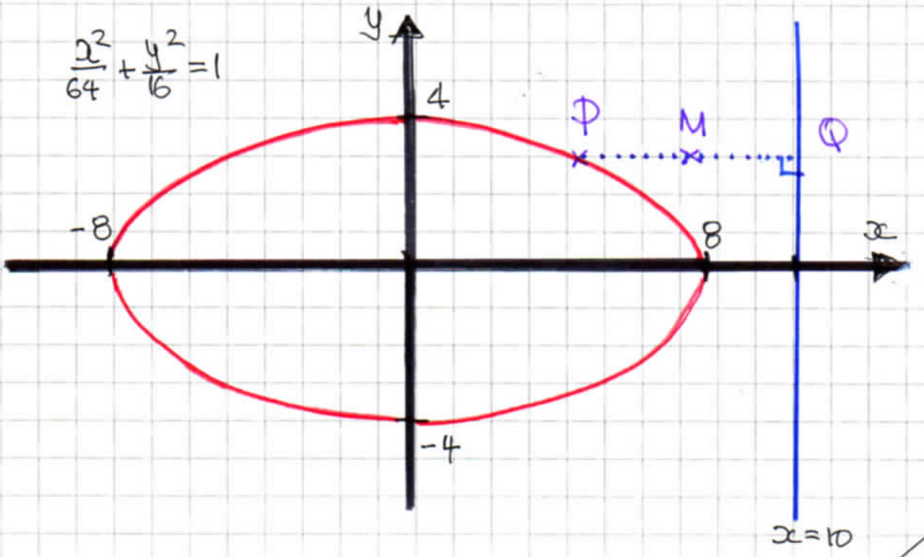
$$= \lim_{x \to 0}\left[\frac{\cancel{1} - 18x^2 + O(x^4) - \cancel{1}}{2x^2}\right]$$

$$= \lim_{x \to 0}\left[-9 + O(x^2)\right]$$

$$= -9$$

IYGB - FP3 PAPER N - QUESTION 8

a) THIS IS A STANDARD ELLIPSE WITH $-8 \leq x \leq 8$, $-4 \leq y \leq 4$



b) PARAMETRIZE THE ELLIPSE

$$x = 8\cos\theta \quad , \quad y = 4\sin\theta \qquad 0 \leq \theta < 2\pi$$

THEN THE CO-ORDINATES OF P, Q & R CAN BE FOUND

- $P(8\cos\theta, 4\sin\theta)$
- $Q(10, 4\sin\theta)$
- $M\left[\frac{8\cos\theta + 10}{2}, \frac{4\sin\theta + 4\sin\theta}{2}\right] = M(5 + 4\cos\theta, 4\sin\theta)$

ELIMINATE THE PARAMETER θ, OUT OF THE GENERAL CO-ORDINATES OF M (WRITTEN AS PARAMETRICS)

$$\left.\begin{array}{l} X = 5 + 4\cos\theta \\ Y = 4\sin\theta \end{array}\right\} \Longrightarrow \begin{array}{l} 4\cos\theta = X - 5 \\ 4\sin\theta = Y \end{array}$$

IYGB - FP3 PAPER N - QUESTION 8

$$\Rightarrow \left.\begin{array}{l} 16\cos^2\theta = (x-5)^2 \\ 16\sin^2\theta = y^2 \end{array}\right\}$$

$$\Rightarrow 16\cos^2\theta + 16\sin^2\theta = (x-5)^2 + y^2$$

$$\Rightarrow 16(\cancel{\cos^2\theta + \sin^2\theta}) = (x-5)^2 + y^2$$

$$\Rightarrow (x-5)^2 + y^2 = 16$$

∴ A CIRCLE, CENTRED AT $(5,0)$, RADIUS 4

IYGB – FP3 PAPER N – QUESTION 9

START BY MANIPULATIONS & AUXILIARIES

- $t = \tan\left(\frac{1}{2}x\right)$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{1}{2}x\right)$$

$$\frac{dt}{dx} = \frac{1}{2}\left[1 + \tan^2\left(\frac{1}{2}x\right)\right]$$

$$\frac{dt}{dx} = \frac{1}{2}\left[1 + t^2\right]$$

$$\frac{dx}{dt} = \frac{2}{1+t^2}$$

$$\boxed{dx = \frac{2}{1+t^2}\,dt}$$

- $x = \frac{\pi}{2} \mapsto t = 1$

$$x = 0 \mapsto t = 0$$

- $t = \tan\frac{1}{2}x$

$$t^2 = \tan^2\frac{1}{2}x$$

$$1 + t^2 = 1 + \tan^2\frac{1}{2}x$$

$$1 + t^2 = \sec^2\frac{1}{2}x$$

$$\frac{1}{1+t^2} = \cos^2\frac{1}{2}x$$

$$\frac{2}{1+t^2} = 2\cos^2\frac{1}{2}x$$

$$\frac{2}{1+t^2} - 1 = 2\cos^2\frac{1}{2}x - 1$$

$$\frac{2-(1+t^2)}{1+t^2} = \cos\left(2\times\frac{1}{2}x\right)$$

$$\boxed{\frac{1-t^2}{1+t^2} = \cos x}$$

TRANSFORMING THE INTEGRAL

$$\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x}\,dx = \int_0^1 \frac{1}{2 - \frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{2(1+t^2) - (1-t^2)}\,dt$$

$$= \int_0^1 \frac{2}{1+3t^2}\,dt$$

IYGB - FP3 PAPER N - QUESTION 9

$$= \frac{2}{3}\int_0^1 \frac{1}{t^2+\frac{1}{3}}\,dt$$

MANIPULATE INTO A STANDARD ARCTAN FORM

$$\dots = \frac{2}{3}\int_0^1 \frac{1}{t^2+\left(\frac{1}{\sqrt{3}}\right)^2}\,dt$$

$$= \frac{2}{3}\times\frac{1}{\frac{1}{\sqrt{3}}}\left[\arctan\left(\frac{1}{\frac{1}{\sqrt{3}}}t\right)\right]_0^1$$

$$= \frac{2}{3}\sqrt{3}\left[\arctan(\sqrt{3}t)\right]_0^1$$

$$= \frac{2}{3}\sqrt{3}\left[\arctan\sqrt{3} - \cancel{\arctan 0}\right]$$

$$= \frac{2}{3}\sqrt{3}\times\frac{\pi}{3}$$

$$= \frac{2\pi\sqrt{3}}{9}$$