IYGB – FP3 QUESTION Q – QUESTION 1

NON GRAPHICAL APPROACH

• IF $x \geq 1$

$|x-1| > 6x-1$

$x-1 > 6x-1$

$-5x > 0$

$x < 0$

$\therefore$ NO SOLUTIONS AS $x \geq 1$

• IF $x \leq 1$

$|x-1| > 6x-1$

$1-x > 6x-1$

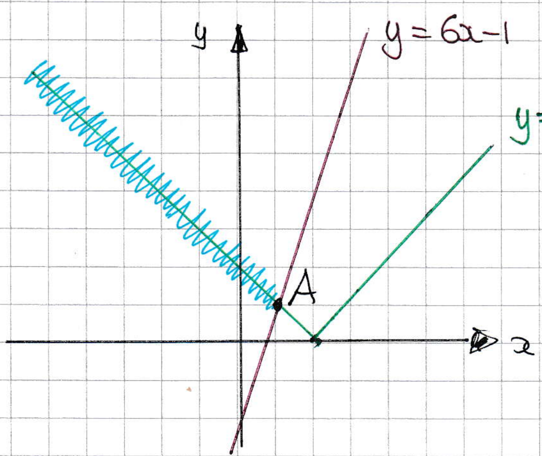
$-7x > -2$

$x < \frac{2}{7}$

VALID SOLUTION INTERVAL

$\therefore$ SOLUTION INTERVAL $x < \frac{2}{7}$

GRAPHICAL APPROACH – SKETCH $y=|x-1|$ & $y=6x-1$



FIND THE INTERSECTION

$6x-1$ = "REFLECTED PART OF $y=|x-1|$

$6x-1 = 1-x$

$7x = 2$

$x = \frac{2}{7}$

$\therefore$ FROM GRAPH

$x < \frac{2}{7}$ // AS ABOVE

IYGB – FP3 PAPER Q – QUESTION 2

a) IF THE VECTORS ARE COPLANAR, THE CROSS PRODUCT OF ANY TWO WILL BE PERPENDICULAR TO THE THIRD

$$\Rightarrow (\underline{a} \wedge \underline{b}) \cdot \underline{c} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & \lambda \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1\begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \lambda\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow (3-6) - 2(1-4) + \lambda(3-6) = 0$$

$$\Rightarrow -3 + 6 - 3\lambda = 0$$

$$\Rightarrow 3 = 3\lambda$$

$$\Rightarrow \lambda = 1$$

b) SETTING UP AN EQUATION

$$\underline{a} = p\underline{b} + q\underline{c}$$

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = p\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + q\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

EQUATE SAY $\underline{i}$ & $\underline{k}$ (THE $\underline{j}$ SHOULD BALANCE)

$$\left.\begin{aligned} 2p + q &= 1 \\ p + q &= 2 \end{aligned}\right\} \Rightarrow p = -1 \quad \& \quad q = 3$$

$$\therefore \underline{a} = 3\underline{c} - \underline{b}$$

IYGB – FP3 PAPER Q – QUESTION 3

a) FIND THE FIRST 3 DERIVATIVES OF $f(x)$

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$

EVALUATE THE FUNCTION AND ITS DERIVATIVES AT $x = \frac{\pi}{6}$

$$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad f'\left(\frac{\pi}{6}\right) = -\frac{1}{2}, \quad f''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \quad f'''\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

BY THE TAYLOR THEOREM

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + O\left[(x-a)^4\right]$$

$$\cos x = \frac{\sqrt{3}}{2} + \left(x-\frac{\pi}{6}\right)\left(-\frac{1}{2}\right) + \frac{\left(x-\frac{\pi}{6}\right)^2}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\left(x-\frac{\pi}{6}\right)^3}{6}\left(\frac{1}{2}\right) + O\left[\left(x-\frac{\pi}{6}\right)^4\right]$$

$$\cos x = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x-\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x-\frac{\pi}{6}\right)^3 + O\left[\left(x-\frac{\pi}{6}\right)^4\right]$$

LET $x = \frac{\pi}{4}$, SO THAT $\left(x - \frac{\pi}{6}\right) = \frac{\pi}{12}$

$$\Rightarrow \cos\frac{\pi}{4} = \frac{\sqrt{3}}{2} - \frac{1}{2}\times\frac{\pi}{12} - \frac{\sqrt{3}}{4}\left(\frac{\pi}{12}\right)^2 + \frac{1}{12}\left(\frac{\pi}{12}\right)^3 + \dots$$

$$\Rightarrow \cos\frac{\pi}{4} = \frac{\sqrt{3}}{2} - \frac{\pi}{24} - \frac{\sqrt{3}\pi^2}{576} + \frac{\pi^3}{20736} + \dots$$

AS REQUIRED

IYGB - FP3 PAPER Q - QUESTION 4

BY L'HOSPITAL RULE SINCE THE LIMIT IS OF THE FORM ZERO OVER ZERO, WE OBTAIN

$$\lim_{x \to 0}\left[\frac{\cos 7x - 1}{x\sin x}\right] = \lim_{x \to 0}\left[\frac{\frac{d}{dx}(\cos 7x - 1)}{\frac{d}{dx}(x\sin x)}\right]$$

$$= \lim_{x \to 0}\left[\frac{-7\sin 7x}{\sin x + x\cos x}\right]$$

THIS AGAIN OF THE TYPE ZERO OVER ZERO, SO RE-APPLY L'HOSPITAL'S RULE

$$= \lim_{x \to 0}\left[\frac{\frac{d}{dx}(-7\sin 7x)}{\frac{d}{dx}(\sin x + x\cos x)}\right]$$

$$= \lim_{x \to 0}\left[\frac{-49\cos 7x}{\cos x + \cos x - x\sin x}\right]$$

$$= -\frac{49}{2}$$

ALTERNATIVE BY SERIES EXPANSIONS

$$\lim_{x \to 0}\left[\frac{\cos 7x - 1}{x\sin x}\right] = \lim_{x \to 0}\left[\frac{\cancel{1} - \frac{(7x)^2}{2!} + O(x^4) - \cancel{1}}{x\left(x - \frac{x^3}{6} + O(x^5)\right)}\right]$$

$$= \lim_{x \to 0}\left[\frac{-\frac{49}{2}x^2 + O(x^4)}{x^2 - \frac{1}{6}x^4 + O(x^6)}\right]$$

$$= \lim_{x \to 0}\left[\frac{-\frac{49}{2} + O(x^2)}{1 - \frac{1}{6}x^2 + O(x^4)}\right]$$

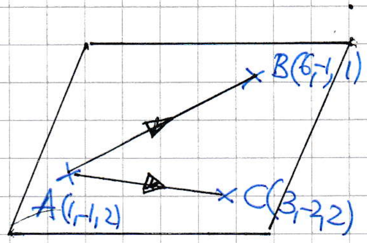
$$= -\frac{49}{2}$$ AS ABOVE

IYGB - FP3 PAPER Q - QUESTION 5

a) START BY FORMING A CROSS PRODUCT

$\vec{AB} = \underline{b} - \underline{a} = (6,-1,1) - (1,-1,2) = (5,0,-1)$

$\vec{AC} = \underline{c} - \underline{a} = (3,-2,2) - (1,-1,2) = (2,-1,0)$



$$\vec{AB} \wedge \vec{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = (-1,-2,-5)$$

TAKE AS A NORMAL TO THE PLANE $(1,2,5)$

$\Longrightarrow 1x + 2y + 5z = \text{CONSTANT}$

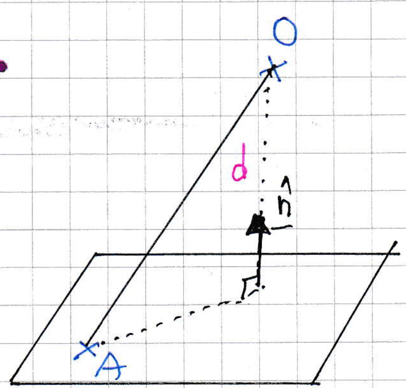
$\Longrightarrow 1 + 2(-1) + 5(2) = \text{CONSTANT}$

(USING $A(1,-1,2)$)

$\Longrightarrow \text{CONSTANT} = 9$

$\therefore \quad x + 2y + 5z = 9 \qquad \underline{r} \cdot (1,2,5) = 9$

b) PROJECT $\vec{OA}$, ONTO THE DIRECTION OF $\hat{n}$



$\Longrightarrow d = |\vec{OA} \cdot \hat{n}|$

$\Longrightarrow d = \left|\vec{OA} \cdot \frac{\underline{n}}{|\underline{n}|}\right|$

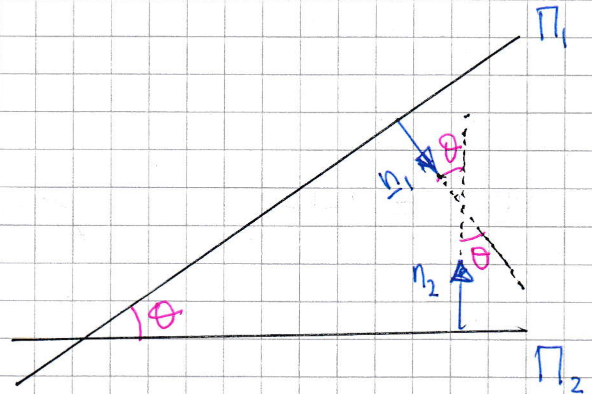
$\Longrightarrow d = \left|\frac{(1,-1,2)\cdot(1,2,5)}{\sqrt{1+4+25}}\right|$

$\Longrightarrow d = \left|\frac{1-2+10}{\sqrt{30}}\right|$

$\Longrightarrow d = \frac{9}{\sqrt{30}} = \frac{3}{10}\sqrt{30}$

IYGB - FP3 PAPER Φ - QUESTION 5

c) LOOKING AT THE CROSS SECTIONS OF THE TWO PLANES & DOTTING THEIR NORMALS



$$\underline{n}_1 \cdot \underline{n}_2 = |\underline{n}_1||\underline{n}_2|\cos\theta$$

$$(1,2,5)\cdot(5,-2,7) = |1,2,5||5,-2,7|\cos\theta$$

$$5-4+35 = \sqrt{1+4+25}\sqrt{25+4+49}\cos\theta$$

$$36 = \sqrt{30}\sqrt{78}\cos\theta$$

$$\cos\theta = \frac{36}{\sqrt{30\times78}}$$

$$\theta \simeq 41.9088\ldots$$

$$\therefore \theta \simeq 42^\circ$$

IYGB – FP3 PAPER Q – QUESTION 6

$$\frac{dy}{dx} = \ln(x+y+1) \qquad x=1,\ y=1$$

WRITE THE O.D.E IN THE USUAL NOTATION

$$y_n' = \ln(x_n + y_n + 1)$$

USE EULER'S FORMULA

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$$

$$y_n' \approx \frac{y_{n+1} - y_n}{h}$$

$$\ln(x_n + y_n + 1) \simeq \frac{y_{n+1} - y_n}{h}$$

$$y_{n+1} \simeq h\ln(x_n + y_n + 1) + y_n \qquad \text{WITH} \quad x_0 = 1,\ y_0 = 1,\ h = 0.1$$

APPLY THE RESULT TWICE

- $y_1 \simeq h\ln(x_0 + y_0 + 1) + y_0$

  $y_1 \simeq 0.1\ln(1+1+1) + 1$

  $y_1 \simeq 1.109861229\ldots$

- $y_2 \simeq h\ln(x_1 + y_1 + 1) + y_1$

  $y_2 \simeq 0.1\ln(1.1 + 1.10986\ldots + 1) + 1.10986\ldots$

  $y_2 \simeq 1.226483999\ldots$

$\therefore\ y \approx 1.226$

IYGB - FP3 PAPER Q - QUESTION 7

USING THE SUBSTITUTION GIVEN

$$t = \tan\left(\frac{x}{2}\right) \implies \frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

$$\frac{dt}{dx} = \frac{1}{2}\left[1 + \tan^2\left(\frac{x}{2}\right)\right]$$

$$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$$

$$2\frac{dt}{dx} = 1+t^2$$

$$dx = \frac{2}{1+t^2}\,dt$$

ALSO USING THE COSINE DOUBLE ANGLE IDENTITY

$$\implies \cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \leftarrow \left\{\cos 2\theta = \cos^2\theta - \sin^2\theta\right\}$$

$$\implies \cos x = \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2$$

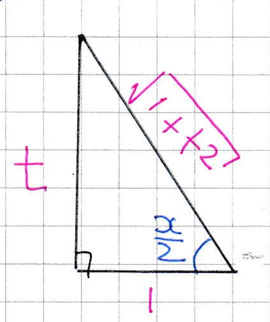
$$\implies \cos x = \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$$

$$\implies \cos x = \frac{1-t^2}{1+t^2}$$

$$\implies 5 + 4\cos x = 5 + \frac{4(1-t^2)}{1+t^2}$$

$$= \frac{5+5t^2+4-4t^2}{1+t^2}$$

$$= \frac{9+t^2}{1+t^2}$$



$\tan\frac{x}{2} = \frac{t}{1}$

(PYTHAGORAS YIELDS A HYPOTENUSE OF $\sqrt{t^2+1}$)

$\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$

$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$

IYGB – FP3 PAPER Q – QUESTION 7

FINALLY THE LIMITS IF $t=\tan\left(\frac{x}{2}\right)$

$$x=0 \longmapsto t=0$$
$$x=\frac{2\pi}{3} \longmapsto t=\sqrt{3}$$

TRANSFORMING THE INTEGRAL

$$\Longrightarrow \int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\,dx = \int_0^{\sqrt{3}} \frac{1}{\frac{9+t^2}{\cancel{1+t^2}}} \times \frac{2}{\cancel{1+t^2}}\,dt$$

$$= \int_0^{\sqrt{3}} \frac{2}{9+t^2}\,dt$$

THIS IS A STANDARD "ARCTAN TYPE" INTEGRAL

$$= \int_0^{\sqrt{3}} \frac{2}{t^2+3^2}\,dt$$

$$= \left[\frac{2}{3}\arctan\left(\frac{t}{3}\right)\right]_0^{\sqrt{3}}$$

$$= \frac{2}{3}\left[\arctan\frac{\sqrt{3}}{3} - \cancel{\arctan 0}\right]$$

$$= \frac{2}{3}\times\frac{\pi}{6}$$

$$= \frac{\pi}{9}$$

-1-

IYGB - FP3 PAPER Q - QUESTION 8

a) OBTAIN THE GRADIENT FUNCTION

$$\frac{d}{dx}\left(\frac{x^2}{16}+\frac{y^2}{4}\right)=\frac{d}{dx}(1)$$

$$\frac{1}{8}x+\frac{1}{2}y\frac{dy}{dx}=0$$

AT THE GENERAL POINT $A(4\cos\theta, 2\sin\theta)$

$$\frac{1}{8}(4\cos\theta)+\frac{1}{2}(2\sin\theta)\left.\frac{dy}{dx}\right|_A=0$$

$$\frac{1}{2}\cos\theta+\left.\frac{dy}{dx}\right|_A\sin\theta=0$$

$$\left.\frac{dy}{dx}\right|_A=-\frac{\cos\theta}{2\sin\theta}$$

OBTAIN THE EQUATION OF THE TANGENT

$$y-2\sin\theta=-\frac{\cos\theta}{2\sin\theta}(x-4\cos\theta)$$

$$2y\sin\theta-4\sin^2\theta=-x\cos\theta+4\cos^2\theta$$

$$2y\sin\theta+x\cos\theta=4(\cos^2\theta+\sin^2\theta)$$

$$2y\sin\theta+x\cos\theta=4$$

AS REQUIRED

b) CIRCLE HAS CENTRE AT $(0,0)$ – GRADIENT OF OB

$$m_{OB}=\frac{4\sin\theta-0}{4\cos\theta-0}=\frac{\sin\theta}{\cos\theta}$$

TANGENT GRADIENT AT B IS $-\frac{\cos\theta}{\sin\theta}$

EQUATION OF TANGENT AT B IS

$$y-4\sin\theta=-\frac{\cos\theta}{\sin\theta}(x-4\cos\theta)$$

$$y\sin\theta-4\sin^2\theta=-x\cos\theta+4\cos^2\theta$$

$$y\sin\theta+x\cos\theta=4(\cos^2\theta+\sin^2\theta)$$

$$y\sin\theta+x\cos\theta=4$$

SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} y\sin\theta+x\cos\theta=4 \\ 2y\sin\theta+x\cos\theta=4 \end{array}\right\}\Rightarrow y\sin\theta=0$$

$$\Rightarrow y=0$$

$$\Rightarrow x\cos\theta=4$$

$$\Rightarrow x=\frac{4}{\cos\theta}$$

$$\therefore P\left(\frac{4}{\cos\theta},0\right)$$

IYGB – FP3 PAPER Q – QUESTION 8

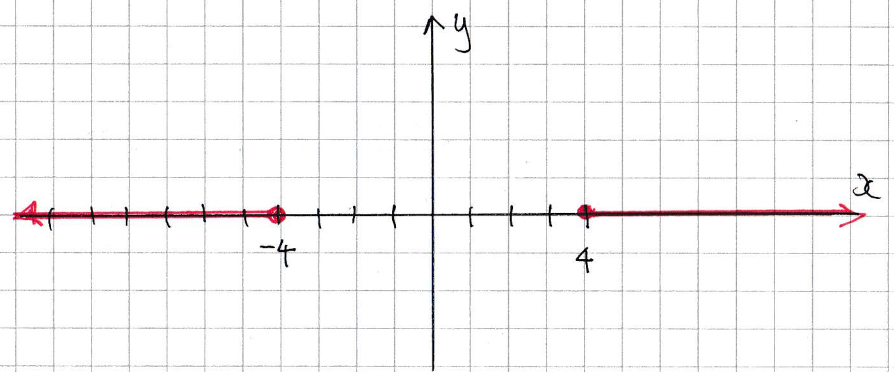
c) THE POINT $P\left(\frac{4}{\cos\theta}, 0\right)$ LIES ON THE x AXIS

$$-1 \leqslant \cos\theta \leqslant 1$$

$$\frac{1}{\cos\theta} \leqslant -1 \quad \cup \quad \frac{1}{\cos\theta} \geqslant 1 \qquad \text{(ESSENTIALLY } \sec\theta\text{)}$$

$$\frac{4}{\cos\theta} \leqslant -4 \quad \cup \quad \frac{4}{\cos\theta} \geqslant 1$$

HENCE THE REQUIRED LOCUS IS $\{y = 0 : \ x \leqslant -4 \cup x \geqslant 4\}$



IYGB - FP3 PAPER Q - QUESTION 9

a) I) DIFFERENTIATING WITH RESPECT TO y

$$t = \tan x \implies \frac{d}{dy}(t) = \frac{d}{dy}(\tan x)$$

$$\implies \frac{dt}{dy} = \sec^2 x \times \frac{dx}{dy}$$

$$\implies \frac{dx}{dy} = \frac{1}{\sec^2 x}\frac{dt}{dy}$$

$$\implies \frac{dy}{dx} = \sec^2 x \frac{dy}{dt}$$

AS REQUIRED

II) NOW DIFFERENTIATING THE ABOVE EXPRESSION WITH RESPECT TO x

$$\implies \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\sec^2 x \frac{dy}{dt}\right)$$

$$\implies \frac{d^2y}{dx^2} = 2\sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$\implies \frac{d^2y}{dx^2} = 2\sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d^2y}{dt^2} \times \frac{dt}{dx}$$

BUT IF $t = \tan x$

$$\frac{dt}{dx} = \sec^2 x$$

$$\implies \frac{d^2y}{dx^2} = 2\sec^2 x \tan x \frac{dy}{dt} + \sec^2 x \frac{d^2y}{dt^2} \sec^2 x$$

$$\implies \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\sec^4 x + 2\frac{dy}{dt}\sec^2 x \tan x$$

AS REQUIRED

c) TRANSFORMING THE GIVEN O.D.E

$$\implies \frac{d^2y}{dx^2} - 2\frac{dy}{dx}\tan x - y\sec^4 x = 0$$

$$\implies \left(\frac{d^2y}{dt^2}\sec^4 x + \cancel{2\frac{dy}{dt}\sec^2 x \tan x}\right) - 2\left(\cancel{\sec^2 x \frac{dy}{dt}}\right)\tan x - y\sec^4 x = 0$$

IYGB - FP3 PAPER Q - QUESTION 9

$$\Longrightarrow \frac{d^2y}{dt^2}\sec^4 x - y\sec^4 x = 0$$

$$\Longrightarrow \frac{d^2y}{dt^2} - y = 0$$

AUXILIARY EQUATION

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$\therefore$ GENERAL SOLUTION IS

$$y = Ae^{t} + Be^{-t} \quad \text{OR} \quad y = P\cosh t + Q\sinh t$$

$$y = Ae^{\tan x} + Be^{-\tan x} \quad \text{OR} \quad y = P\cosh(\tan t) + Q\sinh(\tan t)$$