IYGB – FP4 PAPER M – QUESTION 1

START BY INVESTIGATING CLOSURE

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} c & d \\ d & c \end{pmatrix} = \begin{pmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{pmatrix}$$ IE OF THE SAME FORM

ASSOCIATIVITY – HAS TO BE ASSUMED FROM MATRICES

IF $\underline{A}, \underline{B}$ & $\underline{C}$ ARE 2×2 MATRICES THEN $(\underline{A}\underline{B})\underline{C} \equiv \underline{A}(\underline{B}\underline{C})$

EXISTENCE OF IDENTITY

$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ IS OF THE REQUIRED FORM, AND IF $\underline{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$$\underline{A}\underline{I} = \underline{I}\underline{A} = \underline{A}$$

EXISTENCE OF INVERSE

FOR ALL $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ AN INVERSE EXISTS AS $\frac{1}{a^2-b^2}\begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$

$$= \begin{pmatrix} \frac{a}{a^2-b^2} & \frac{b}{b^2-a^2} \\ \frac{b}{b^2-a^2} & \frac{a}{a^2-b^2} \end{pmatrix}$$

WHICH AGAIN IS OF THE CORRECT FORM

THEREFORE INDEED A GROUP

IYGB - FP4 PAPER M - QUESTION 2

SETTING UP EUCLID'S ALGORITHM FOR 560 & 1169

$$1169 = 2 \times 560 + 49$$
$$560 = 11 \times 49 + 21$$
$$49 = 2 \times 21 + 7 \quad \leftarrow \text{HCF}$$
$$21 = 3 \times 7 + 0$$

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IYGB - FP4 PAPER 4 - QUESTION 3

$$t_{n+1} = 3t_n + 2, \quad t_1 = 1, \quad n \in \mathbb{N} \iff t_n = 2 \times 3^{n-1} - 1, \quad n \in \mathbb{N}$$

BASE CASE

$$\left.\begin{aligned} t_1 &= 1 \\ t_1 &= 2 \times 3^{1-1} - 1 = 2 \times 1 - 1 = 1 \end{aligned}\right\} \text{RESULT HOLDS FOR } n = 1$$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$

$$\Longrightarrow t_k = 2 \times 3^{k-1} - 1$$

$$\Longrightarrow 3t_k = 3\left[2 \times 3^{k-1} - 1\right]$$

$$\Longrightarrow 3t_k = 2 \times 3^k - 3$$

$$\Longrightarrow 3t_k + 2 = 2 \times 3^k - 3 + 2$$

$$\Longrightarrow t_{k+1} = 2 \times 3^{(k+1)-1} - 1$$

CONCLUSION

IF THE RESULT HOLDS FOR $n = k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n = k+1$

SINCE THE RESULT HOLDS FOR $n = 1$, THEN IT MUST HOLD FOR ALL n

IYGB – FP4 PAPER M – QUESTION 4

PRELIMINARIES FIRST

$$\Rightarrow y = \ln(1+\cos x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin x}{\cos x + 1}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-\sin x}{\cos x+1}\right)^2 = 1 + \frac{\sin^2 x}{(1+\cos x)^2} = \frac{(1+\cos x)^2 + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{1 + 2\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)} = \frac{2+2\cos x}{(1+\cos x)^2} = \frac{2(1+\cos x)}{(1+\cos x)^2}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{2}{1+\cos x} \qquad (\cos x \neq -1)$$

SETTING AN ARCLENGTH INTEGRAL

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{2}{1+\cos x}}\, dx = 2\int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{1+\cos x}}\, dx$$

USING DOUBLE ANGLE IDENTITIES

$$s = 2\int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{1+(2\cos^2\frac{x}{2}-1)}}\, dx = 2\int_0^{\frac{\pi}{2}} \sqrt{\frac{\cancel{2}}{\cancel{2}\cos^2\frac{x}{2}}}\, dx$$

$$s = 2\int_0^{\frac{\pi}{2}} \frac{1}{\cos\frac{x}{2}}\, dx = 2\int_0^{\frac{\pi}{2}} \sec\frac{x}{2}\, dx = 2\Big[2\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right|\Big]_0^{\frac{\pi}{2}}$$

STANDARD RESULT

$$s = 4\left[\ln\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \ln(\sec 0 + \tan 0)\right] = 4\left[\ln(\sqrt{2}+1) - \cancel{\ln(1+0)}\right]$$

$$s = 4\ln(\sqrt{2}+1) = 2\ln(\sqrt{2}+1)^2 = 2\ln(2+2\sqrt{2}+1) = 2\ln(3+2\sqrt{2})$$

REPEATING ONCE MORE

$$s = \ln(3+2\sqrt{2})^2 = \ln(9+12\sqrt{2}+8) = \ln(17+12\sqrt{2})$$

AS REQUIRED

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a) PROCEED BY INTEGRATION BY PARTS

$$\Rightarrow I_n = \int_0^{\pi/2} x^n \cos x \, dx$$

x^n	nx^{n-1}
$\sin x$	$\cos x$

$$\Rightarrow I_n = \left[x^n \sin x\right]_0^{\pi/2} - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n \times 1 - 0 - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x$$

INTEGRATION BY PARTS AGAIN

x^{n-1}	$(n-1)x^{n-2}$
$-\cos x$	$\sin x$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n\left[\underbrace{\cancel{\left[-x^{n-1}\cos x\right]_0^{\pi/2}}}_{\text{ZERO AT BOTH ENDS}} + (n-1)\int_0^{\pi/2} x^{n-2} \cos x \, dx\right]$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)\int_0^{\pi/2} x^{n-2} \cos x \, dx$$

$$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$$

AS REQUIRED

b) I) REWRITE IN "I" NOTATION

$$\int_0^{\pi/2} x^4 \cos x \, dx = I_4$$

$$= \left(\frac{\pi}{2}\right)^4 - 4\times 3\, I_2 = \frac{\pi^4}{16} - 12 I_2$$

$$= \frac{\pi^4}{16} - 12\left[\left(\frac{\pi}{2}\right)^2 - 2\times 1\times I_0\right]$$

$$= \frac{\pi^4}{16} - 12\times\frac{\pi^2}{4} + 24 I_0$$

$$= \frac{\pi^4}{16} - 3\pi^2 + 24\int_0^{\pi/2} \cos x \, dx$$

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$$= \frac{\pi^4}{16} - 3\pi^2 + 24\Big[\sin x\Big]_0^{\pi/2}$$

$$= \frac{\pi^4}{16} - 3\pi^2 + 24$$

II) START BY INTEGRATION BY PARTS

$$\int_0^{\frac{\pi}{2}} x^5 \sin x = \cancel{\Big[-x^5\cos x\Big]_0^{\pi/2}} - \int -5x^4 \cos x \, dx$$

ZERO AT BOTH ENDS

x^5	$5x^4$
$-\cos x$	$\sin x$

$$= 5\int x^4 \cos x \, dx$$

$$= 5 I_4$$

$$= 5\left(\frac{\pi^4}{16} - 3\pi^2 + 24\right)$$

$$= \frac{5}{16}\pi^4 - 15\pi^2 + 120$$

IYGB - FP4 PAPER N - QUESTION 6

PROCEED AS FOLLOWS

$$|z+1-i| = |z-1+2i| \quad \text{WITH} \quad f(z) = kz+i$$

$$w = kz+i$$

$$\frac{w-i}{k} = z$$

SUBSTITUTE INTO THE LINE & TIDY

$$\Longrightarrow \left|\frac{w-i}{k}+1+i\right| = \left|\frac{w-i}{k}-1+2i\right|$$

$$\Longrightarrow \frac{1}{k}\left|w-i+k+ik\right| = \frac{1}{k}\left|w-i-k+2ki\right|$$

LET $w = x+iy$

$$\Longrightarrow |x+iy-i+k+ik| = |x+iy-i-k+2ki|$$

$$\Longrightarrow |(x+k)+i(y-1+k)| = |(x-k)+i(y-1+2k)|$$

$$\Longrightarrow \sqrt{(x+k)^2+(y-1+k)^2} = \sqrt{(x-k)^2+(y-1+2k)^2}$$

$$\Longrightarrow \cancel{x^2}+2kx+\cancel{k^2}+\cancel{y^2}+\cancel{1}+k^2\cancel{-2y}-2k+2yk = \cancel{x^2}-2kx+\cancel{k^2}+\cancel{y^2}+\cancel{1}+4k^2\cancel{-2y}-4k+4ky$$

$$\Longrightarrow 2kx+k^2-2k+2ky = -2kx+4k^2-4k+4ky$$

$$\Longrightarrow 4kx-3k^2+2k = 2ky$$

$$\Longrightarrow y = 2x+1-\frac{3}{2}k$$

$$\therefore \quad 1-\frac{3}{2}k = -8$$

$$9 = \frac{3}{2}k$$

$$\underline{k=6}$$

IYGB - FP4 PAPER M - QUESTION 7

a)
$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & -3 & 1 \\ 3 & -5 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \therefore \lambda = 2$$

b)
$$\left.\begin{aligned} x - y + z &= -2x \\ 3x - 3y + z &= -2y \\ 3x - 5y + 3z &= -2z \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 3x - y + z &= 0 \\ 3x - y + z &= 0 \\ 3x - 5y + 5z &= 0 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} y &= 3x + z \\ 3x - 5(3x+z) + 5z &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} y &= 3x + z \\ -12x &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} y &= z \\ x &= 0 \end{aligned}$$

$$\therefore \underline{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

c)
$$\underline{A}^7\underline{w} = \underline{A}^7(\underline{u} + \underline{v}) = \underline{A}^7\underline{u} + \underline{A}^7\underline{v}$$

$$= \underline{A}^6[\underline{A}\underline{u} + \underline{A}\underline{v}] = \underline{A}^6[2\underline{u} - 2\underline{v}]$$

$$= \underline{A}^5[\underline{A}\,2\underline{u} - \underline{A}\,2\underline{v}] = \underline{A}^5[4\underline{u} + 4\underline{v}]$$

$$= \underline{A}^4[\underline{A}\,4\underline{u} + \underline{A}\,4\underline{v}] = \underline{A}^4[8\underline{u} - 8\underline{v}]$$

$$\vdots \qquad\qquad \vdots$$

$$= \underline{A}[\underline{A}\,2^5\underline{u} + \underline{A}(-2)^5\underline{v}] = \underline{A}[2^6\underline{u} + (-2)^6\underline{v}]$$

$$= 2^7\underline{u} + (-2)^7\underline{v} = 128\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 128\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 128 \\ 128 \\ 256 \end{pmatrix} - \begin{pmatrix} 0 \\ 128 \\ 128 \end{pmatrix} = \begin{pmatrix} 128 \\ 0 \\ 128 \end{pmatrix}$$

IYGB – FP4 PAPER M – QUESTION 8

a) REQUIRED NUMBER IS GIVEN BY

$$\frac{7!}{3!\,2!} = 420$$

TRIPLE "A" (REPEAT)   DOUBLE "N" (REPEAT)

b) TREATING THE VOWELS "AAA", AS A SINGLE LETTER

$\boxed{AAA}_1$ $\boxed{B}_2$ $\boxed{N}_3$ $\boxed{N}_4$ $\boxed{S}_5$

$$\frac{5!}{2!} = 60$$

DOUBLE "N" REPEAT

c) BLOCKING THE VOWELS & CONSONANTS TOGETHER

$\boxed{A\ A\ A}$ $\boxed{B\ N\ N\ S}$

1 WAY $\frac{4!}{2!}$ WAYS = 12

HENCE $(1 \times 12) \times 2 = 24$

3 VOWELS – 4 CONSONANTS OR 4 CONSONANTS – 3 VOWELS

IYGB - FP4 PAPER M - QUESTION 8

d) SPLITTING IN SEPARATE CASES & NOTE IN THIS PART ORDER DOES NOT MATTER

I $\underline{A}\,\underline{A}\,\underline{A}$ WITH ONE OF B, N, S 3 WAYS

II $\underline{A}\,\underline{A}$ WITH TWO OF B, N, S 3 WAYS

III $\underline{N}\,\underline{N}$ WITH TWO OF A, B, S 3 WAYS

IV $\underline{A}\,\underline{A}\,\underline{N}\,\underline{N}$ 1 WAY

V ALL DIFFERENT B A N S 1 WAY

11 WAYS

e) USING PART (d)

CASE I, SAY A A A B $3 \times \frac{4!}{3!} = 12$

CASE II, SAY A A B N $3 \times \frac{4!}{2!} = 36$

CASE III, SAY N N A B $3 \times \frac{4!}{2!} = 36$

CASE IV, A A N N $1 \times \frac{4!}{2!\,2!} = 6$

CASE V B A N S $1 \times 4! = 24$

114