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LYGB - FP4 PAPER M - QUESTION 1

START BY INVESTIGATING CLOSURE

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} c & d \\ d & c \end{pmatrix} = \begin{pmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{pmatrix} \text{ IS OF THE SAME FORM}$$

ASSOCIATIVITY - HAS TO BE ASSUMED FROM MATRICES

$$\text{IF } \underline{A}, \underline{B} \text{ \& } \underline{C} \text{ ARE } 2 \times 2 \text{ MATRICES THEN } (\underline{AB})\underline{C} \equiv \underline{A}(\underline{BC})$$

EXISTENCE OF IDENTITY

$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ IS OF THE REQUIRED FORM, AND IF } \underline{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$\underline{AI} = \underline{IA} = \underline{A}$$

EXISTENCE OF INVERSE

$$\begin{aligned} \text{FOR ALL } \begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{ AN INVERSE EXISTS AS } & \frac{1}{a^2-b^2} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix} \\ & = \begin{pmatrix} \frac{a}{a^2-b^2} & \frac{b}{b^2-a^2} \\ \frac{b}{b^2-a^2} & \frac{a}{a^2-b^2} \end{pmatrix} \end{aligned}$$

WHICH AGAIN IS OF
THE CORRECT FORM

THEREFORE IT IS A GROUP

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SETTING UP EUCLID'S ALGORITHM FOR 560 & 1169

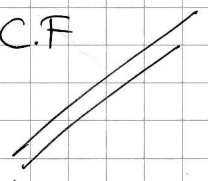
$$1169 = 2 \times 560 + 49$$

$$560 = 11 \times 49 + 21$$

$$49 = 2 \times 21 + 7$$

$$21 = 3 \times 7 + 0$$

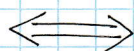
← H.C.F



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IYGB - FP4 PAPER 4 - QUESTION 3

$$\begin{aligned}t_{n+1} &= 3t_n + 2 \\ t_1 &= 1 \\ n &\in \mathbb{N}\end{aligned}$$



$$\begin{aligned}t_n &= 2 \times 3^{n-1} - 1 \\ n &\in \mathbb{N}\end{aligned}$$

BASE CASE

$$t_1 = 1$$

$$t_1 = 2 \times 3^{1-1} - 1 = 2 \times 1 - 1 = 1$$

} RESULT HOLDS FOR $n=1$

INDUCTIVE HYPOTHESIS

SUPPOSE THAT THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$

$$\Rightarrow t_k = 2 \times 3^{k-1} - 1$$

$$\Rightarrow 3t_k = 3[2 \times 3^{k-1} - 1]$$

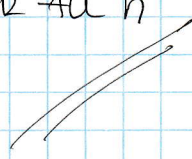
$$\Rightarrow 3t_k = 2 \times 3^k - 3$$

$$\Rightarrow 3t_k + 2 = 2 \times 3^k - 3 + 2$$

$$\Rightarrow t_{k+1} = 2 \times 3^{(k+1)-1} - 1$$

CONCLUSION

IF THE RESULT HOLDS FOR $n=k$, $k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$
SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n



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LYGB - FP4 PAPER M - QUESTION 4

PRELIMINARIES FIRST

$$\Rightarrow y = \ln(1 + \cos x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin x}{\cos x + 1}$$

$$\begin{aligned}\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{-\sin x}{\cos x + 1}\right)^2 = 1 + \frac{\sin^2 x}{(\cos x + 1)^2} = \frac{(1 + \cos x)^2 + \sin^2 x}{(\cos x + 1)^2} \\ &= \frac{1 + 2\cos x + \cos^2 x + \sin^2 x}{(\cos x + 1)^2} = \frac{2 + 2\cos x}{(\cos x + 1)^2} = \frac{2(1 + \cos x)}{(\cos x + 1)^2}\end{aligned}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{2}{1 + \cos x} \quad (\cos x \neq -1)$$

SETTING AN ARCLENGTH INTEGRAL

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{2}{1 + \cos x}} dx = 2 \int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{1 + \cos x}} dx$$

USING DOUBLE ANGLE IDENTITIES

$$s = 2 \int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{1 + (2\cos^2 \frac{x}{2} - 1)}} dx = 2 \int_0^{\frac{\pi}{2}} \sqrt{\frac{2}{2\cos^2 \frac{x}{2}}} dx$$

$$s = 2 \int_0^{\frac{\pi}{2}} \frac{1}{\cos \frac{x}{2}} dx = 2 \int_0^{\frac{\pi}{2}} \sec \frac{x}{2} dx = 2 \left[\ln \left| \sec \frac{x}{2} + \tan \frac{x}{2} \right| \right]_0^{\frac{\pi}{2}}$$

STANDARD RESULT

$$s = 4 \left[\ln \left(\sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right) - \ln \left(\sec 0 + \tan 0 \right) \right] = 4 \left[\ln(\sqrt{2} + 1) - \ln(1 + 0) \right]$$

$$s = 4 \ln(\sqrt{2} + 1) = 2 \ln(\sqrt{2} + 1)^2 = 2 \ln(2 + 2\sqrt{2} + 1) = 2 \ln(3 + 2\sqrt{2})$$

REPEATING ONCE MORE

$$s = \ln(3 + 2\sqrt{2})^2 = \ln(9 + 12\sqrt{2} + 8) = \ln(17 + 12\sqrt{2})$$

AS REQUIRED

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a) PROCEED BY INTEGRATION BY PARTS

$$\Rightarrow I_n = \int_0^{\pi/2} x^n \cos x \, dx$$

$$\Rightarrow I_n = [x^n \sin x]_0^{\pi/2} - n \int_0^{\pi/2} x^{n-1} \sin x \, dx$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n \times 1 - 0 - n \int_0^{\pi/2} x^{n-1} \sin x \, dx$$

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n \int_0^{\pi/2} x^{n-1} \sin x \, dx$$

x^n	$n x^{n-1}$
$\sin x$	$\cos x$

INTEGRATION BY PARTS AGAIN

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n \left[\cancel{[-x^{n-1} \cos x]_0^{\pi/2}} + (n-1) \int_0^{\pi/2} x^{n-2} \cos x \, dx \right]$$

ZERO AT BOTH ENDS

$$\Rightarrow I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) \int_0^{\pi/2} x^{n-2} \cos x \, dx$$

$$\therefore I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) \int_0^{\pi/2} x^{n-2} \cos x \, dx$$

x^{n-1}	$(n-1)x^{n-2}$
$-\cos x$	$\sin x$

AS REQUIRED

b) i) REWRITE IN "I" NOTATION

$$\begin{aligned} \int_0^{\pi/2} x^4 \cos x \, dx &= I_4 \\ &= \left(\frac{\pi}{2}\right)^4 - 4 \times 3 I_2 = \frac{\pi^4}{16} - 12 I_2 \\ &= \frac{\pi^4}{16} - 12 \left[\left(\frac{\pi}{2}\right)^2 - 2 \times 1 \times I_0 \right] \\ &= \frac{\pi^4}{16} - 12 \times \frac{\pi^2}{4} + 24 I_0 \\ &= \frac{\pi^4}{16} - 3\pi^2 + 24 \int_0^{\pi/2} \cos x \, dx \end{aligned}$$

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$$= \frac{\pi^4}{16} - 3\pi^2 + 24 \left[\sin x \right]_0^{\pi/2}$$

$$= \underline{\underline{\frac{\pi^4}{16} - 3\pi^2 + 24}}$$

II START BY INTEGRATION BY PARTS

$$\int_0^{\pi/2} x^5 \sin x = \left[-x^5 \cos x \right]_0^{\pi/2} - \int -5x^4 \cos x dx$$

zero at both ends

x^5	$\sin x$
$- \cos x$	$\sin x$

$$= 5 \int x^4 \cos x dx$$

$$= 5 I_4$$

$$= 5 \left(\frac{\pi^4}{16} - 3\pi^2 + 24 \right)$$

$$= \underline{\underline{\frac{5}{16} \pi^4 - 15\pi^2 + 120}}$$

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1YGB - FPL PAGE 11 - QUESTION 6

PROCEED AS FOLLOWS

$$|z+1-i| = |z-1+2i| \quad \text{with} \quad \begin{aligned} f(z) &= kz+i \\ w &= kz+i \\ \frac{w-i}{k} &= z \end{aligned}$$

SUBSTITUTE INTO THE LINE & TIDY

$$\begin{aligned} \Rightarrow \left| \frac{w-i}{k} + 1 + i \right| &= \left| \frac{w-i}{k} - 1 + 2i \right| \\ \Rightarrow \frac{1}{k} |w-i+k+ik| &= \frac{1}{k} |w-i-k+2ki| \end{aligned}$$

LET $w = x+iy$

$$\Rightarrow |x+iy-i+k+ik| = |x+iy-i-k+2ki|$$

$$\Rightarrow |(x+k) + i(y-1+k)| = |(x-k) + i(y-1+2k)|$$

$$\Rightarrow \sqrt{(x+k)^2 + (y-1+k)^2} = \sqrt{(x-k)^2 + (y-1+2k)^2}$$

$$\Rightarrow \cancel{x^2} + 2kx + \cancel{k^2} + \cancel{y^2} + 1 + \cancel{k^2} - 2y - 2k + 2yk = \cancel{x^2} - 2kx + \cancel{k^2} + \cancel{y^2} + 1 + 4k^2 - 2y - 4k + 4ky$$

$$\Rightarrow 2kx + k^2 - 2k + 2ky = -2kx + 4k^2 - 4k + 4ky$$

$$\Rightarrow 4kx - 3k^2 + 2k = 2ky$$

$$\Rightarrow y = 2x + 1 - \frac{3}{2}k$$

$$\therefore 1 - \frac{3}{2}k = -8$$

$$9 = \frac{3}{2}k$$

$$k = 6$$

LYGB - FP4 PAPER M - QUESTION 7

$$a) \begin{pmatrix} 1 & -1 & 1 \\ 3 & -3 & 1 \\ 3 & -5 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \therefore \lambda = 2$$

$$b) \left. \begin{array}{l} x - y + z = -2x \\ 3x - 3y + z = -2y \\ 3x - 5y + 3z = -2z \end{array} \right\} \Rightarrow \left. \begin{array}{l} 3x - y + z = 0 \\ 3x - y + z = 0 \\ 3x - 5y + 5z = 0 \end{array} \right\} \Rightarrow$$

$$\left. \begin{array}{l} y = 3x + z \\ 3x - 5(3x + z) + 5z = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} y = 3x + z \\ -12x = 0 \end{array} \right\} \Rightarrow \begin{array}{l} y = z \\ x = 0 \end{array}$$

$$\therefore \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$c) \underline{A} \underline{w} = \underline{A} (\underline{u} + \underline{v}) = \underline{A}^7 \underline{u} + \underline{A}^7 \underline{v}$$

$$= \underline{A}^6 [\underline{A} \underline{u} + \underline{A} \underline{v}] = \underline{A}^6 [2\underline{u} - 2\underline{v}]$$

$$= \underline{A}^5 [\underline{A} 2\underline{u} - \underline{A} 2\underline{v}] = \underline{A}^5 [4\underline{u} + 4\underline{v}]$$

$$= \underline{A}^4 [\underline{A} 4\underline{u} + \underline{A} 4\underline{v}] = \underline{A}^4 [8\underline{u} - 8\underline{v}]$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$= \underline{A} [\underline{A} 2^5 \underline{u} + \underline{A} (-2)^5 \underline{v}] = \underline{A} [2^6 \underline{u} + (-2)^6 \underline{v}]$$

$$= 2^7 \underline{u} + (-2)^7 \underline{v} = 128 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 128 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 128 \\ 128 \\ 256 \end{pmatrix} - \begin{pmatrix} 0 \\ 128 \\ 128 \end{pmatrix} = \begin{pmatrix} 128 \\ 0 \\ 128 \end{pmatrix}$$

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NYGB - FP4 PAPER M - QUESTION 8

a) REQUIRED NUMBER IS GIVEN BY

$$\frac{7!}{3! \cdot 2!} = 420$$

↑ ↑
TRIPLE "A"
(REPEAT) DOUBLE "N"
(REPEAT)

b) TREATING THE VOWELS "AAA", AS A SINGLE LETTER

AAA	B	N	N	S
1	2	3	4	5

$$\frac{5!}{2!} = 60$$

↑
DOUBLE "N" REPEAT

c) BLOCKING THE VOWELS & CONSONANTS TOGETHER

AAA	B N N S
1 WAY	$\frac{4!}{2!}$ WAYS = 12

$$\text{#Ways } (1 \times 12) \times 2 = 24$$

↑
3 VOWELS - 4 CONSONANTS OR 4 CONSONANTS - 3 VOWELS

1YGB - FP4 PAPER 11 - QUESTION 8

d) SPUTTING IN SEPARATE CASES & NOTE IN THIS PART ORDER DOES NOT MATTER

I A A A WITH ONE OF B, N, S 3 WAYS

II A A WITH TWO OF B, N, S 3 WAYS

III N N WITH TWO OF A, B, S 3 WAYS

IV A A N N 1 WAY

V ALL DIFFERENT B A N S 1 WAY

11 WAYS

e) USING PART (d)

CASE I, SAY A A A B $3 \times \frac{4!}{3!} = 12$

CASE II, SAY A A B N $3 \times \frac{4!}{2!} = 36$

CASE III, SAY N N A B $3 \times \frac{4!}{2!} = 36$

CASE IV, A A N N $1 \times \frac{4!}{2! \cdot 2!} = 6$

CASE V B A N S $1 \times 4! = 24$

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