IYGB - FURTHER SYNOPTIC PAPER A - QUESTION 1

APPLY THE CROSS PRODUCT FIRST

$$\Longrightarrow \quad \underline{a} \cdot [\underline{b} \wedge (\underline{c} + \underline{a})] = \underline{a} \cdot [\underline{b} \wedge \underline{c} + \underline{b} \wedge \underline{a}]$$

$$= \underline{a} \cdot \underline{b} \wedge \underline{c} + \underline{a} \cdot \underline{b} \wedge \underline{a}$$

NOW $\underline{b} \wedge \underline{a}$ IS PERPENDICULAR TO $\underline{a}$, SO $\underline{a} \cdot (\underline{b} \wedge \underline{a}) = 0$

$$\therefore \quad \underline{a} \cdot [\underline{b} \wedge (\underline{c} + \underline{a})] = \underline{a} \cdot \underline{b} \wedge \underline{c} \quad //$$

IYGB - SYNF PAPER 4 - QUESTION 2

MANIPULATE AS FOLLOWS

$$\tan 2x + \tan 4x = 0$$

$$\tan 4x = -\tan 2x$$

$$\tan 4x = \tan(-2x)$$

HENCE WE NOW HAVE

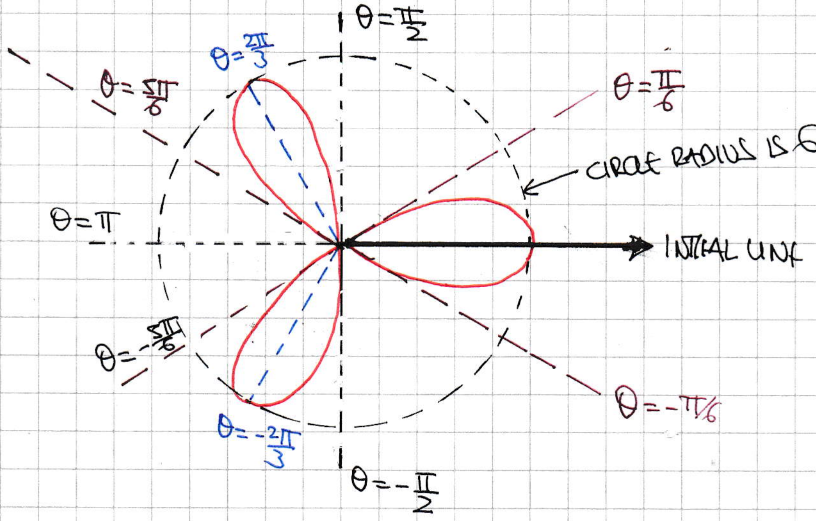
$$4x = -2x \pm n\pi \qquad n = 0, 1, 2, 3, \ldots$$

$$6x = \pm n\pi$$

$$x = \pm \frac{1}{6} n\pi$$

IYGB - FURTHER SYNOPTIC PAPER A - QUESTION 3

a) THIS IS A "TRI-FOIL" WITH MAXIMUM r OCCURING AT $\theta = 0, \frac{2\pi}{3}, -\frac{2\pi}{3}$



b) THIS IS THE AREA OF ONE PETAL

$$\text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r(\theta))^2 \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (6\cos 3\theta)^2 \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 36\cos^2 3\theta \, d\theta$$

EVEN INTEGRAND

$$= \int_0^{\pi/6} 36\cos^2 3\theta \, d\theta = \int_0^{\frac{\pi}{6}} 36\left(\frac{1}{2} + \frac{1}{2}\cos 6\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{6}} 18 + 18\cos 6\theta \, d\theta = \Big[18\theta + 3\sin 6\theta\Big]_0^{\frac{\pi}{6}}$$

$$= (3\pi + 0) - (0 + 0) = 3\pi$$

IYGB - SYNF PAPER A - QUESTION 4

a) DETERMINE THE MODULUS & ARGUMENT OF -16

$|-16| = 16 \qquad \arg(-16) = \pi$

THUS WE NOW HAVE

$$\Rightarrow z^4 = -16 = 16e^{i(\pi + 2k\pi)} \qquad k \in \mathbb{Z}$$

$$\Rightarrow z^4 = 16e^{i\pi(1+2k)}$$

$$\Rightarrow (z^4)^{\frac{1}{4}} = \left[16e^{i\pi(2k+1)}\right]^{\frac{1}{4}}$$

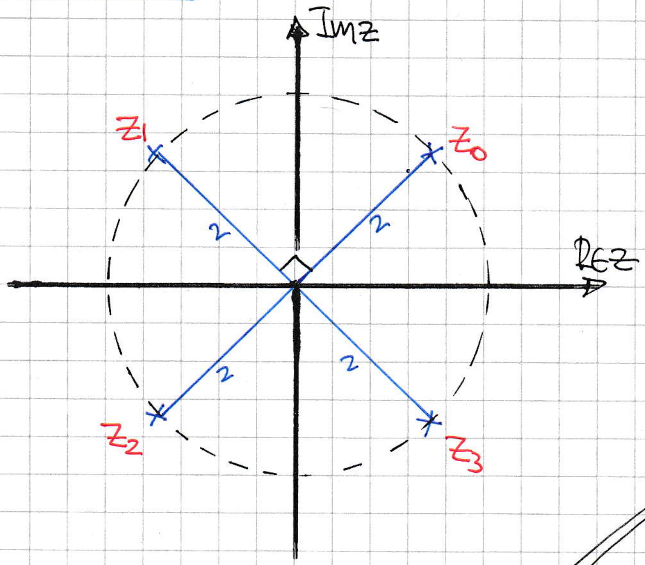
$$\Rightarrow z = 16^{\frac{1}{4}} e^{i\frac{\pi}{4}(2k+1)}$$

$$\Rightarrow z = 2e^{i\frac{\pi}{4}(2k+1)}$$

SUBSTITUTING $k = 0, 1, 2, 3$

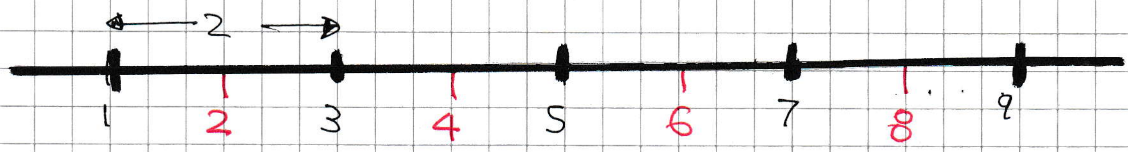
- $z_0 = 2e^{i\frac{\pi}{4}} = 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = 2\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] = \sqrt{2} + \sqrt{2}\,i$
- $z_1 = 2e^{i\frac{3\pi}{4}} = 2\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = 2\left[-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] = -\sqrt{2} + \sqrt{2}\,i$
- $z_2 = 2e^{\frac{5\pi}{4}i} = 2\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = 2\left[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right] = -\sqrt{2} - \sqrt{2}\,i$
- $z_3 = 2e^{\frac{7\pi}{4}i} = 2\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = 2\left[\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right] = \sqrt{2} - \sqrt{2}\,i$

b) FINALLY THE SKETCH



IYGB – FURTHER SYNOPTIC PAPER A – QUESTION 5

DRAWING A TABLE OF VALUES BASED ON MIDPOINTS



x	2	4	6	8
$\ln(1+x^3)$	$\ln 9$	$\ln 65$	$\ln 217$	$\ln 513$

USING THE MID-ORDINATE RULE

AREA $\simeq$ (THICKNESS) $\times$ (SUM OF ALL)

$\simeq 2 \times (\ln 9 + \ln 65 + \ln 217 + \ln 513)$

$\simeq 35.98357\ldots$

$\simeq 36.0$ // 3 sf

IYGB - SYNF PAPER A - QUESTION 6

DIFFERENTIATE & SET EQUAL TO ZERO

$$\Rightarrow y = 5 - 12x + 4\operatorname{arcosh} 4x$$

$$\Rightarrow \frac{dy}{dx} = -12 + 4 \times \frac{4}{\sqrt{16x^2 - 1}}$$

$$\Rightarrow 0 = -12 + \frac{16}{\sqrt{16x^2 - 1}}$$

$$\Rightarrow 12 = \frac{16}{\sqrt{16x^2 - 1}}$$

$$\Rightarrow \sqrt{16x^2 - 1} = \frac{4}{3}$$

$$\Rightarrow 16x^2 - 1 = \frac{16}{9}$$

$$\Rightarrow 16x^2 = \frac{16}{9} + 1$$

$$\Rightarrow 16x^2 = \frac{25}{9}$$

$$\Rightarrow x^2 = \frac{25}{144}$$

$$\Rightarrow x = +\frac{5}{12}$$

(OTHERWISE arcosh IS NOT DEFINED FOR NEGATIVE)

NOW SUBSTITUTE INTO THE EQUATION

$$y = 5 - 12 \times \frac{5}{12} + 4\operatorname{arcosh}\left(4 \times \frac{5}{12}\right)$$

$$y = \cancel{5 - 5} + 4\operatorname{arcosh}\left(\frac{5}{3}\right)$$

$$y = 4\ln\left(\frac{5}{3} + \sqrt{\left(\frac{5}{3}\right)^2 - 1}\right)$$

$$y = 4\ln\left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1}\right)$$

$$y = 4\ln\left(\frac{5}{3} + \frac{4}{3}\right)$$

$$y = 4\ln 3$$

IYGB - FURTHER SYNOPTIC PAPER A - QUESTION 7

a) START BY OBTAINING THE GRADIENT FUNCTION FOLLOWED BY THE TANGENT

$$y^2 = 36x$$

$$2y\frac{dy}{dx} = 36$$

$$\frac{dy}{dx} = \frac{18}{y}$$

$$\left.\frac{dy}{dx}\right|_{y=18t} = \frac{18}{18t} = \frac{1}{t}$$

$$\Rightarrow y - y_0 = m(x - x_0)$$

$$\Rightarrow y - 18t = \frac{1}{t}(x - 9t^2)$$

$$\Rightarrow yt - 18t^2 = x - 9t^2$$

$$\Rightarrow 0 = x - ty + 9t^2$$

$$\Rightarrow x - ty + 9t^2 = 0$$

AS REQUIRED

b) AT $Q(1,6)$ WE NEED THE VALUE OF t

$$\Rightarrow 18t = 6$$

$$\Rightarrow t = \frac{1}{3}$$

EQUATION OF THE TANGENT, WHERE $t = \frac{1}{3}$

$$\Rightarrow x - \frac{1}{3}y + 9\left(\frac{1}{3}\right)^2 = 0$$

$$\Rightarrow x - \frac{1}{3}y + 1 = 0$$

$$\Rightarrow 3x - y + 3 = 0$$

FINALLY FIND THE EQUATION OF THE DIRECTRIX

$$y^2 = 36x = 4(9x) \quad \text{i.e. } "a = 9" \Rightarrow \text{DIRECTRIX } x = -9$$

$$\Rightarrow 3x - y + 3 = 0$$

$$\Rightarrow -27 - y + 3 = 0$$

$$\Rightarrow -24 = y$$

$$\therefore D(-9, -24)$$

IYGB – FURTHER SYNOPTIC PAPER A – QUESTION 8

- FIRSTLY IF THERE IS NO UNIQUE SOLUTION, THE DETERMINANT OF THE MATRIX $\begin{bmatrix} 1 & 2 & 1 \\ 2 & a & 1 \\ 1 & 1 & 2 \end{bmatrix}$ MUST BE ZERO

- EXPAND BY TOP ROW

$$1\begin{vmatrix} a & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & a \\ 1 & 1 \end{vmatrix} = 0$$

$$(2a-1) - (2\times3) + (2-a) = 0$$

$$2a - 1 - 6 + 2 - a = 0$$

$$a - 5 = 0$$

$$a = 5$$

- START ROW REDUCING TO OBTAIN A "BOTTOM ZERO ROW" IF THE SYSTEM IS TO BE CONSISTENT

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 5 & 1 & 2 \\ 1 & 1 & 2 & b \end{array}\right] \begin{array}{c} r_{12}(-2) \\ r_{13}(-1) \end{array} = \left[\begin{array}{ccc|c} 0 & 2 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & -1 & 1 & b-2 \end{array}\right]$$

$$r_{23}(1) = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & b-4 \end{array}\right] \qquad \therefore \ b = 4$$

- CONTINUING ROW REDUCING, IGNORING THE BOTTOM ROW

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -2 \end{array}\right] r_{21}(-2) = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

IYGB - FURTHER SYNOPTIC PAPER A - QUESTION 8

- EXTRACTING THE SOLUTION, WE HAVE

$$x + 3z = 6$$
$$y - z = -2$$

- LET $z = t$, SOME PARAMETER

$$x = 6 - 3t$$
$$y = -2 + t$$
$$z = t$$

IE $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

OR $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6-3t \\ t-2 \\ t \end{pmatrix}$ //

IYGB - FURTHER SYNOPTIC PAPER A - QUESTION 9

PROCEED AS FOLLOWS

$$\frac{d}{dx}\left[Ae^{x}\sin x + Be^{x}\cos x\right] \equiv e^{x}(2\cos x - 3\sin x)$$

$$\frac{d}{dx}\left[e^{x}(A\sin x + B\cos x)\right] \equiv e^{x}(2\cos x - 3\sin x)$$

$$e^{x}(A\sin x + B\cos x) + e^{x}(A\cos x - B\sin x) \equiv e^{x}(2\cos x - 3\sin x)$$

$$(A-B)\sin x + (B+A)\cos x \equiv 2\cos x - 3\sin x$$

SOLVING TWO SIMPLE EQUATIONS

$$\left.\begin{aligned} A-B &= -3 \\ A+B &= 2 \end{aligned}\right\} \text{ ADD \& SUBTRACT}$$

$$2A = -1 \qquad \& \qquad 2B = 5$$

$$A = -\frac{1}{2} \qquad\qquad B = \frac{5}{2}$$

HENCE WE HAVE

$$e^{x}(2\cos x - 3\sin x) = \frac{d}{dx}\left[-\frac{1}{2}e^{x}\sin x + \frac{5}{2}e^{x}\cos x\right]$$

$$\int e^{x}(2\cos x - 3\sin x)\,dx = -\frac{1}{2}e^{x}\sin x + \frac{5}{2}e^{x}\cos x + C$$

$$\int e^{x}(2\cos x - 3\sin x)\,dx = \frac{1}{2}e^{x}\left[5\cos x - \sin x\right] + C$$

IYGB - SYNF PAPER A - QUESTION 10

BERNOULLI INEQUALITY

$$(1+a)^n > 1+an \qquad a \in \mathbb{R},\ a>-1$$
$$n \in \mathbb{N},\ n \geq 2$$

PROOF BY INDUCTION

- IF $n=2$ $\quad LHS = (1+a)^2 = a^2+2a+1$

  $RHS = 1+2a$

  $\therefore\ a^2+2a+1 > 2a+1$, SO THE RESULT HOLDS FOR $n=2$

- SUPPOSE THAT THE INEQUALITY HOLDS FOR $n=k \in \mathbb{N}$, $k>2$

$$\Rightarrow (1+a)^k > 1+ak$$

$$\Rightarrow (1+a)^k(1+a) > (1+a)(1+ak)$$

$$\Rightarrow (1+a)^{k+1} > 1+ak+a+a^2k$$

$$\Rightarrow (1+a)^{k+1} > 1+a(k+1)+\underbrace{a^2k}_{\text{(POSITIVE)}} > 1+a(k+1)$$

$$\Rightarrow (1+a)^{k+1} > 1+a(k+1)$$

- IF THE INEQUALITY HOLDS FOR $n=k\in\mathbb{N}$, $k>2$, THEN IT WILL ALSO HOLD FOR $n=k+1$.

  AS THE INEQUALITY HOLDS FOR $n=2$, THEN IT MUST HOLD FOR ALL POSITIVE INTEGERS GREATER THAN 2

IYGB – SYNF PAPER A – QUESTION 11

GET ALL THE THREE VALUES OF THE SUMS

$$x^3 + 0x^2 + 2x - 1 = 0$$

$$\boxed{\begin{aligned} \alpha+\beta+\gamma &= -\tfrac{0}{1} = 0 \\ \alpha\beta+\beta\gamma+\gamma\alpha &= +\tfrac{2}{1} = 2 \\ \alpha\beta\gamma &= -\tfrac{-1}{1} = 1 \end{aligned}}$$

START THE TIDY UP

$$\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} = \frac{\beta^4\gamma^4 + \alpha^4\gamma^4 + \alpha^4\beta^4}{(\alpha\beta\gamma)^4} = \frac{(\beta^2\gamma^2)^2 + (\alpha^2\gamma^2)^2 + (\alpha^2\beta^2)^2}{(\alpha\beta\gamma)^4}$$

NOW USING $A^2 + B^2 + C^2 \equiv (A+B+C)^2 - 2(AB+BC+CA)$

$$\ldots = \frac{(\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2)^2 - 2(\alpha^2\beta^2\gamma^4 + \alpha^4\beta^2\gamma^2 + \alpha^2\beta^4\gamma^2)}{1^4}$$

$$= (\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2)^2 - 2\alpha^2\beta^2\gamma^2(\gamma^2 + \alpha^2 + \beta^2)$$

$$= (\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2)^2 - 2(\alpha\beta\gamma)^2(\alpha^2 + \beta^2 + \gamma^2)$$

$$= \left[(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2\right]^2 - 2 \times 1^2 (\alpha^2 + \beta^2 + \gamma^2)$$

$$= \left[(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2\right]^2 - 2(\alpha^2 + \beta^2 + \gamma^2)$$

REAPPLY THE IDENTITY FROM ABOVE

$$= \left[(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2(\alpha\beta^2\gamma + \alpha\beta\gamma^2 + \alpha^2\beta\gamma)\right]^2 - 2\left[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\right]$$

$$= \left[(\alpha\beta+\beta\gamma+\gamma\alpha)^2 - \cancel{2\alpha\beta\gamma(\alpha+\beta+\gamma)}\right]^2 - 2\left[\cancel{(\alpha+\beta+\gamma)^2} - 2(\alpha\beta+\beta\gamma+\gamma\alpha)\right]$$

$$= (\alpha\beta+\beta\gamma+\gamma\alpha)^4 + 4(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$= 2^4 + 4 \times 2$$

$$= 24$$

IYGB - SYN F PAPER A - QUESTION 12

a) STANDARD PARTIAL FRACTION METHODOLOGY

$$\frac{2t}{(t+1)(t^2+1)} \equiv \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$$

$$2t \equiv A(t^2+1) + (t+1)(Bt+C)$$
$$2t \equiv At^2+A + Bt^2+Ct+Bt+C$$
$$2t \equiv (A+B)t^2+(B+C)t+(A+C)$$

- IF $t=-1$: $-2=2A$, $A=-1$
- $A+B=0$, $-1+B=0$, $B=1$
- $A+C=0$, $-1+C=0$, $C=1$

b) USING THE LITTLE "t IDENTITIES" VIA THE SUBSTITUTION $t=\tan\frac{x}{2}$, AND QUOTING ALL RESULTS AS THIS IS A STANDARD "SET UP"

$$t=\tan\frac{x}{2} \Rightarrow dx = \frac{2}{1+t^2}\,dt \quad \& \quad \sin x = \frac{2t}{1+t^2}$$
$$\cos x = \frac{1-t^2}{1+t^2}$$

THUS WE CAN TRANSFORM THE INTEGRAL INCLUDING THE LIMITS

$$\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\cos x}{1+\sin x}}\,dx = \int_0^1 \sqrt{\frac{1-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}}} \times \frac{2}{1+t^2}\,dt$$

MULTIPLY "TOP & BOTTOM" BY $(1+t^2)$

$$= \int_0^1 \sqrt{\frac{1+t^2-1+t^2}{1+t^2+2t}}\left(\frac{2}{1+t^2}\right)dt$$

$$= \int_0^1 \sqrt{\frac{2t^2}{(t+1)^2}}\left(\frac{2}{1+t^2}\right)dt = \ldots$$ NO NEED FOR MODULI FOR THESE LIMITS

IYGB – SYNF PAPER A – QUESTION 12

$$= \int_0^1 \frac{2\sqrt{2}\,t}{(t+1)(t^2+1)}\,dt = \sqrt{2}\int_0^1 \frac{2t}{(t+1)(t^2+1)}\,dt$$

USING PART (a)

$$= \sqrt{2}\int_0^1 \frac{t+1}{t^2+1} - \frac{1}{t+1}\,dt = \sqrt{2}\int_0^1 \frac{t}{t^2+1} + \frac{1}{t^2+1} - \frac{1}{t+1}\,dt$$

$$= \sqrt{2}\left[\tfrac{1}{2}\ln(t^2+1) + \arctan t - \ln(t+1)\right]_0^1$$

$$= \sqrt{2}\left[\left(\tfrac{1}{2}\ln 2 + \tfrac{\pi}{4} - \ln 2\right) - \left(\tfrac{1}{2}\cancel{\ln 1} + 0 - \cancel{\ln 1}\right)\right]$$

$$= \sqrt{2}\left(\tfrac{\pi}{4} - \tfrac{1}{2}\ln 2\right) = \frac{\sqrt{2}}{4}(\pi - 2\ln 2)$$

IYGB – FURTHER SYNOPTIC PAPER A – QUESTION 13

a) WRITE THE LINE IN FULL PARAMETRIC FORM

$$\underline{r} = \begin{pmatrix} -2 \\ -12 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \lambda-2 \\ 3\lambda-12 \\ 2\lambda-9 \end{pmatrix}$$

- $\lambda - 2 = a$
- $3\lambda - 12 = b$
- $2\lambda - 9 = 3$

$2\lambda = 12$

$\lambda = 6$

- $3 \times 6 - 12 = b$

$b = 6$

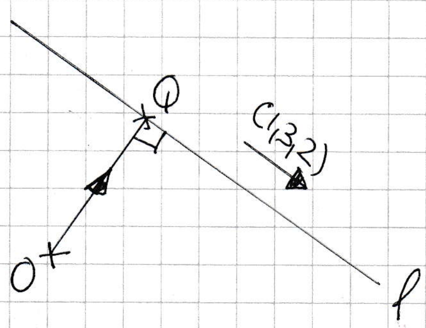
- $6 - 2 = a$

$a = 4$

b) LOOKING AT THE DIAGRAM, LET $Q = (x, y, z)$

$\overrightarrow{OQ} \perp l \Rightarrow (x,y,z) \cdot (1,3,2) = 0$

$\Rightarrow \boxed{x + 3y + 2z = 0}$



BUT $Q(x,y,z)$ LIES ON l

$$\Rightarrow \boxed{\begin{aligned} x &= \lambda - 2 \\ y &= 3\lambda - 12 \\ z &= 2\lambda - 9 \end{aligned}}$$

$\Rightarrow (\lambda - 2) + 3(3\lambda - 12) + 2(2\lambda - 9) = 0$

$\Rightarrow \lambda - 2 + 9\lambda - 36 + 4\lambda - 18 = 0$

$\Rightarrow 14\lambda = 56$

$\Rightarrow \lambda = 4$

$\therefore Q(4-2, 3\times4-12, 2\times4-9)$

$Q(2, 0, -1)$

IYGB - FURTHER SYNOPTIC PAPER A - QUESTION 13

c) LOOKING AT THE DIAGRAM

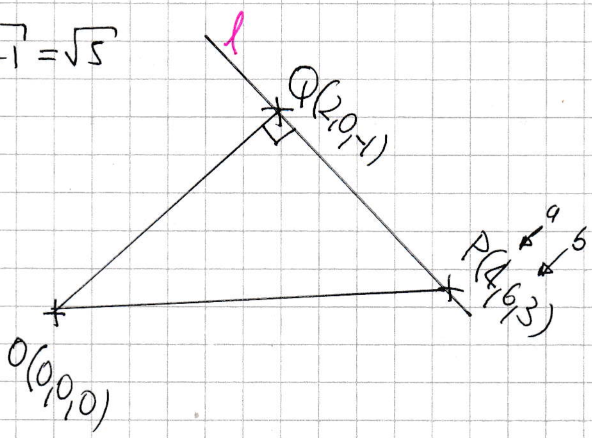
$|\vec{OQ}| = |q| = |2,0,-1| = \sqrt{4+0+1} = \sqrt{5}$

$|\vec{PQ}| = |q - p| = |(2,0,-1)-(4,6,3)|$

$= |-2,-6,-4|$

$= \sqrt{4+36+16}$

$= \sqrt{56}$



$\therefore$ AREA $= \frac{1}{2}|OQ||PQ| = \frac{1}{2}\sqrt{5} \times \sqrt{56} = \frac{1}{2}\sqrt{280}$

$= \sqrt{70}$

IYGB – FURTHER SYNOPTIC PAPER A – QUESTION 14

START WITH THE AUXILIARY EQUATION, TO FIND THE COMPLEMENTARY FUNCTION

$$\frac{d^2y}{dx^2} + y = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$\therefore$ COMPLEMENTARY FUNCTION

$$y = A\cos x + B\sin x$$

FOR PARTICULAR INTEGRAL, AS THE $\frac{dy}{dx}$ IS MISSING, TRY $y = P\sin 2x$

$$\left.\begin{aligned} y &= P\sin 2x \\ \frac{dy}{dx} &= 2P\cos 2x \\ \frac{d^2y}{dx^2} &= -4P\sin 2x \end{aligned}\right\}$$

SUB INTO THE O.D.E

$$-4P\sin 2x + P\sin 2x \equiv \sin 2x$$

$$-3P = 1$$

$$P = -\frac{1}{3}$$

$\therefore$ GENERAL SOLUTION IS

$$y = A\cos x + B\sin x - \frac{1}{3}\sin 2x$$

DIFFERENTIATE & APPLY BOUNDARY CONDITION

$$\frac{dy}{dx} = -A\sin x + B\cos x - \frac{2}{3}\cos 2x$$

- $x = \frac{\pi}{2},\ y = 0 \implies 0 = 0 + B + 0$

$$B = 0$$

IYGB – FURTHER SYNOPTIC PAPER A – QUESTION 14

- $x = \frac{\pi}{2}, \frac{dy}{dx} = 0 \quad \Rightarrow 0 = -A + 0 + \frac{2}{3}$

$\Rightarrow A = \frac{2}{3}$

FINALLY WE HAVE

$$y = \frac{2}{3}\cos x - \frac{1}{3}\sin 2x$$

$$y = \frac{2}{3}\cos x - \frac{2}{3}\sin x\cos x$$

$$y = \frac{2}{3}\cos x(1-\sin x)$$

// AS REQUIRED

IYGB – SYNF PAPER A – QUESTION 15

METHOD A

PICK 3 "EASY" POINTS WHICH SATISFY THE EQUATION OF THE PLANE

$$2x+3y+4z=24 \implies A(12,0,0) \quad B(0,8,0) \quad C(0,0,6)$$

TRANSFORM THESE THREE POINTS

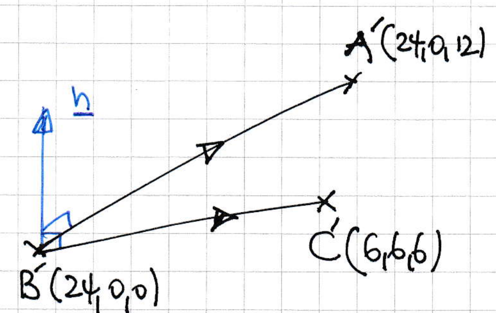
$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 12 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 24 & 24 & 6 \\ 0 & 0 & 6 \\ 12 & 0 & 6 \end{pmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ A & B & C \end{array} \qquad \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ A' & B' & C' \end{array}$$

NOW LOOKING AT THE DIAGRAM

$$\vec{B'A'} = (24,0,12)-(24,0,0) = (0,0,12)$$

$$\vec{B'C'} = (6,6,6)-(24,0,0) = (-18,6,6)$$



SCALE THESE VECTORS & FIND COMMON PERPENDICULAR

$(0,0,1)$ & $(-3,1,1)$ ARE MUTUALLY PERPENDICULAR TO $\underline{n}=(a,b,c)$

- $(0,0,1)\cdot(a,b,c)=0 \implies c=0$
- $(-3,1,1)\cdot(a,b,c)=0 \implies -3a+b+c=0$

$$\implies -3a+b=0$$
$$\implies b=3a$$

(WE COULD HAVE USED THE CROSS PRODUCT HERE)

$$\therefore \underline{n} = (1,3,0)$$

IYGB - SYNF PAPER A - QUESTION 15

HENCE THE EQUATION OF THE TRANSFORMED PLANE IS

$$x + 3y = \text{CONSTANT}$$

& USING ANY OF A', B', C' WE FIND

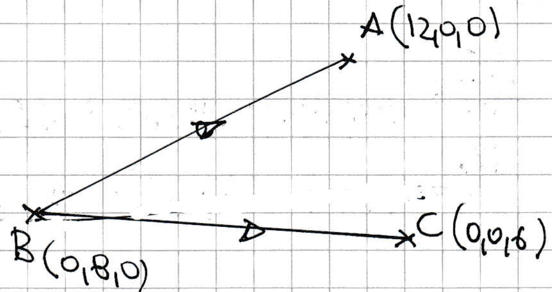
$$x + 3y = 24$$

METHOD B

$$2x + 3y + 4z = 24$$

TAKE 3 RANDOM POINTS ON THIS PLANE AS BEFORE

$A(12,0,0)$, $B(0,8,0)$, $C(0,0,6)$



$\overrightarrow{BA} = \underline{a} - \underline{b} = (12,0,0) - (0,8,0) = (12,-8,0)$ SCALE TO $(3,-2,0)$

$\overrightarrow{BC} = \underline{c} - \underline{b} = (0,0,6) - (0,8,0) = (0,-8,6)$ SCALE TO $(0,-4,3)$

OBTAIN THE PARAMETRIC EQUATION OF THIS PLANE

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3\lambda \\ 8 - 2\lambda - 4\mu \\ 3\mu \end{pmatrix}$$

IYGB - SYNF PAPER A - QUESTION 15

TRANSFORM USING THE MATRIX

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3\lambda \\ 8-2\lambda-4\mu \\ 3\mu \end{pmatrix} = \begin{pmatrix} \cancel{6\lambda}+24-\cancel{6\lambda}-12\mu+3\mu \\ 3\mu \\ 3\lambda+3\mu \end{pmatrix} = \begin{pmatrix} 24-9\mu \\ 3\mu \\ 3\lambda+3\mu \end{pmatrix}$$

ELIMINATING THE PARAMETRICS

$$x = 24-9\mu$$
$$y = 3\mu \quad \Longrightarrow \quad x = 24-3y$$
$$z = 3\lambda+3\mu$$

$$\therefore \ x+3y=24$$

AS BEFORE

IYGB – SYNF PAPER A – QUESTION 16

a) $y = \arcsin x \quad -1 \leqslant x \leqslant 1 \implies -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$

MAKE x THE SUBJECT AND DIFFERENTIATE WITH RESPECT TO y

$$\implies \sin y = x$$
$$\implies x = \sin y$$
$$\implies \frac{dx}{dy} = \cos y$$
$$\implies \frac{dx}{dy} = \pm\sqrt{1-\sin^2 y}$$

AS $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ $\quad \cos y \geqslant 0$

$\pm\sqrt{1-\sin^2 y} \geqslant 0$

$$\implies \frac{dx}{dy} = +\sqrt{1-\sin^2 y}$$
$$\implies \frac{dx}{dy} = \sqrt{1-x^2}$$
$$\implies \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

AS REQUIRED

b) DIFFERENTIATING IMPLICITLY WITH RESPECT TO x

$$\frac{d}{dx}(\arcsin 3x) + \frac{d}{dx}(2\arcsin y) = \frac{d}{dx}\left(\frac{\pi}{2}\right)$$

$$\frac{1}{\sqrt{1-(3x)^2}} \times 3 + 2 \times \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\frac{3}{\sqrt{1-9x^2}} + \frac{2}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

NEXT DETERMINE THE VALUE OF k

$$\implies \arcsin\left(3 \times \frac{1}{6}\right) + 2\arcsin k = \frac{\pi}{2}$$
$$\implies \arcsin\left(\frac{1}{2}\right) + 2\arcsin k = \frac{\pi}{2}$$

IYGB - SYNF PAPER A - QUESTION 16

$$\Rightarrow \frac{\pi}{6} + 2\arcsin k = \frac{\pi}{2}$$

$$\Rightarrow 2\arcsin k = \frac{\pi}{3}$$

$$\Rightarrow \arcsin k = \frac{\pi}{6}$$

$$\Rightarrow k = \frac{1}{2}$$

FINALLY WE HAVE THE GRADIENT AT $P\left(\frac{1}{6}, \frac{1}{2}\right)$

$$\Rightarrow \frac{3}{\sqrt{1-9x^2}} + \frac{2}{\sqrt{1-y^2}} \left.\frac{dy}{dx}\right|_P = 0$$

$$\Rightarrow \frac{3}{\sqrt{1-\frac{1}{4}}} + \frac{2}{\sqrt{1-\frac{1}{4}}} \left.\frac{dy}{dx}\right|_P = 0$$

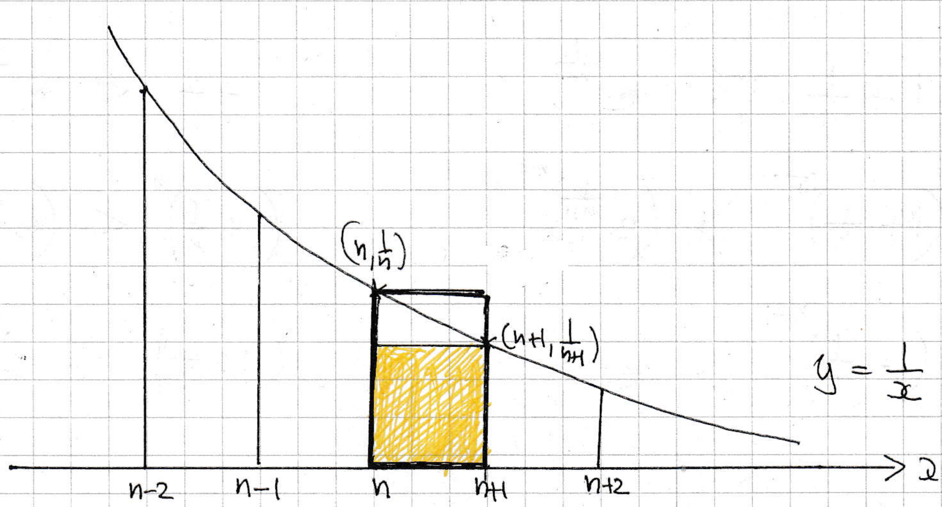
$$\Rightarrow \frac{3}{\sqrt{3/4}} + \frac{2}{\sqrt{3/4}} \left.\frac{dy}{dx}\right|_P = 0$$

$$\Rightarrow 3 + 2\left.\frac{dy}{dx}\right|_P = 0$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_P = -\frac{3}{2}$$

IYGB - SYNOPTIC FURTHER MATHS PAPER A - QUESTION 17

LOOKING AT THE DIAGRAM BELOW, WITH n = INTEGER



THE AREA UNDER THE CURVE BETWEEN n & $n+1$ IS

- GREATER THAN THE YELLOW RECTANGLE OF AREA $1 \times \frac{1}{n+1}$
- LESS THAN THE "BOLD" RECTANGLE OF AREA $1 \times \frac{1}{n}$

HENCE

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n}$$

$$\frac{1}{n+1} < \Big[\ln x\Big]_n^{n+1} < \frac{1}{n}$$

$$\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$$

$$\frac{1}{n+1} < \ln\left(\frac{n+1}{n}\right) < \frac{1}{n}$$

$$e^{\frac{1}{n+1}} < \frac{n+1}{n} < e^{\frac{1}{n}}$$

$$e^{\frac{1}{n+1}} < 1 + \frac{1}{n} < e^{\frac{1}{n}}$$

IYGB – SYNOPTIC FURTHER MATHS PAPER A – QUESTION 17

DEALING WITH EACH PART OF THE INEQUALITY SEPARATELY

$$1+\frac{1}{n} > e^{\frac{1}{n+1}} \qquad\qquad 1+\frac{1}{n} < e^{\frac{1}{n}}$$

$$\left(1+\frac{1}{n}\right)^{n+1} > \left(e^{\frac{1}{n+1}}\right)^{n+1} \qquad\qquad \left(1+\frac{1}{n}\right)^{n} < \left(e^{\frac{1}{n}}\right)^{n}$$

$$\left(1+\frac{1}{n}\right)^{n+1} > e \qquad\qquad \left(1+\frac{1}{n}\right)^{n} < e$$

COMBINING WE OBTAIN

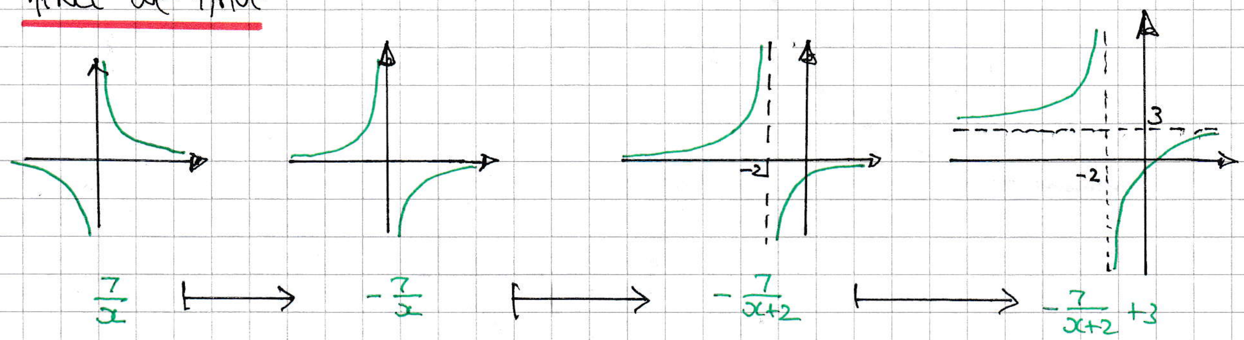
$$\left(1+\frac{1}{n}\right)^{n} < e < \left(1+\frac{1}{n}\right)^{n+1}$$

IYGB - SYNOPTIC FURTHER PAPER A - QUESTION 18

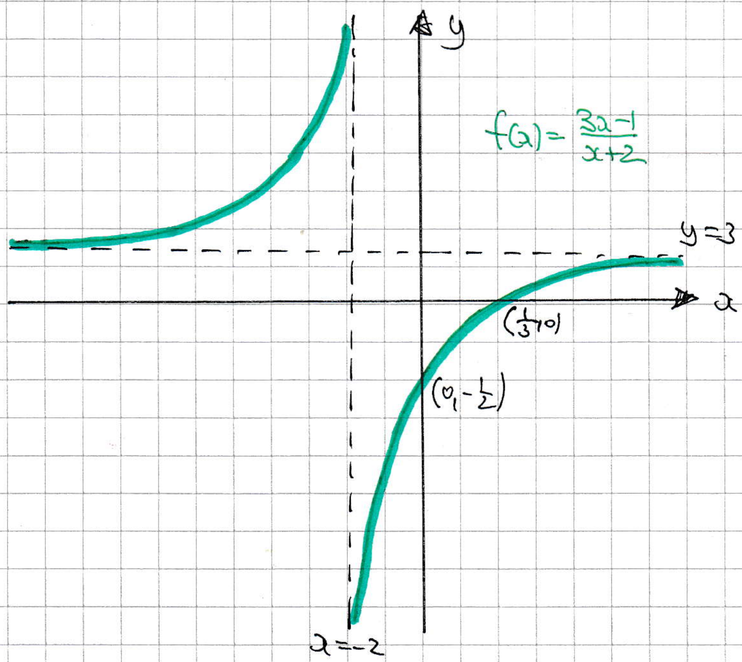
a) START REWRITING $f(x)$ IN ORDER TO "SEE" THE TRANSFORMATIONS

$$f(x) = \frac{3x-1}{x+2} = \frac{3(x+2)-7}{x+2} = 3 - \frac{7}{x+2}$$

HENCE WE HAVE



COMPLETING THE GRAPH



- $x=0$

  $y = \frac{0-1}{0+2} = -\frac{1}{2}$

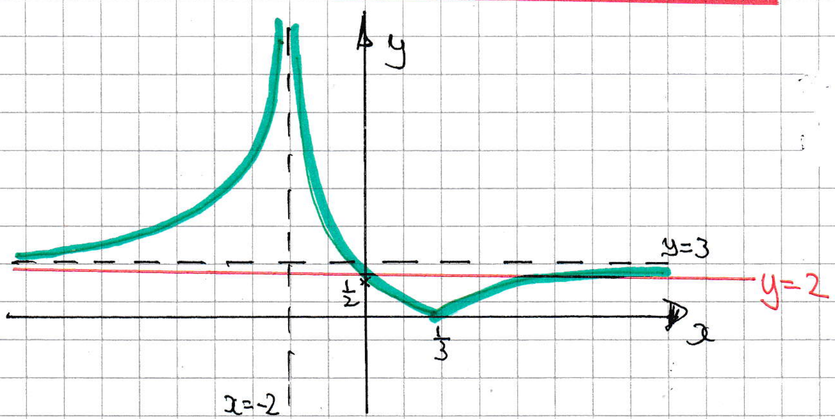
- $y=0$

  $\frac{3x-1}{x+2} = 0$

  $3x-1=0$

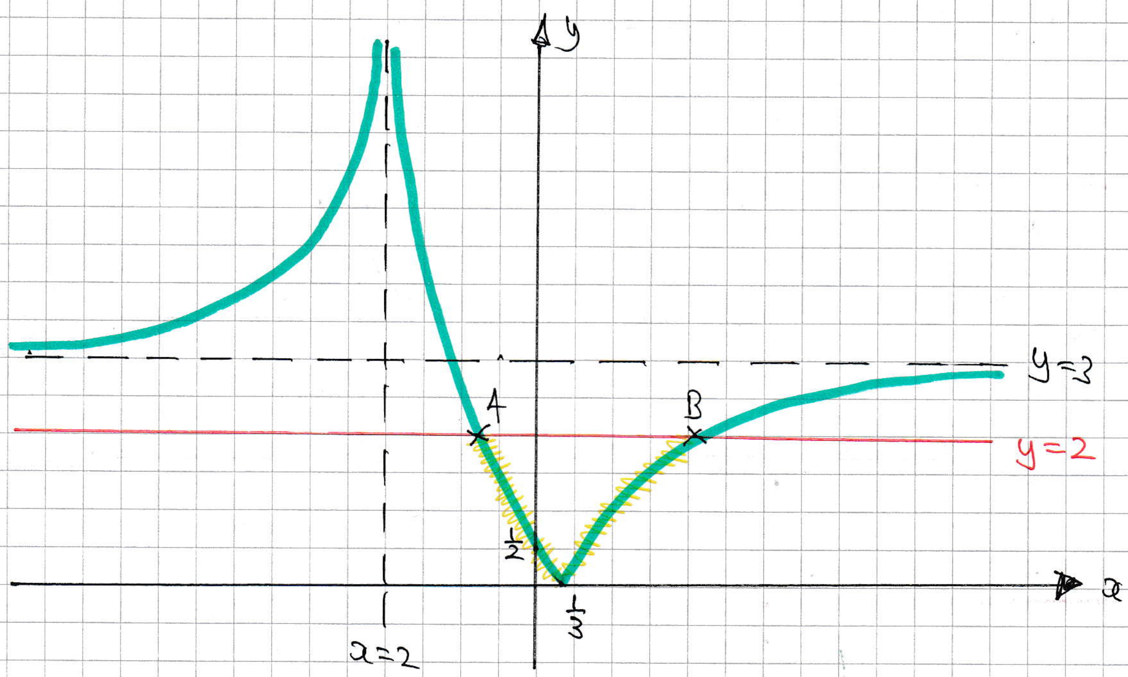
  $x = \frac{1}{3}$

b) PROCEED TO SKETCH THE GRAPH OF $y = |f(x)|$



IYGB - SYNF - PAPER A - QUESTION 18

REDRAWN TO BETTER SCALE



TO FIND A

$$-\left(\frac{3x-1}{x+2}\right)=2$$

$$\frac{3x-1}{x+2}=-2$$

$$3x-1=-2x-4$$

$$5x=-3$$

$$x=-\frac{3}{5}$$

TO FIND B

$$\frac{3x-1}{x+2}=2$$

$$3x-1=2x+4$$

$$x=5$$

$\therefore$ FROM GRAPH $\qquad -\frac{3}{5}<x<5$

IYGB - SYNF PAPER A - QUESTION 19

a) WRITE THE O.D.E IN THE "USUAL FORM" AND LOOK FOR AN INTEGRATING FACTOR

$$\Rightarrow \frac{dy}{dx} = x - 2y$$

$$\Rightarrow \frac{dy}{dx} + 2y = x$$

$$\text{I.F.} = e^{\int 2\,dy} = e^{2y}$$

$$\Rightarrow \frac{d}{dx}\left(ye^{2x}\right) = xe^{2x}$$

$$\Rightarrow ye^{2x} = \int xe^{2x}\,dx$$

INTEGRATION BY PARTS IN THE R.H.S.

x	1
$\frac{1}{2}e^{2x}$	e^{2x}

$$\Rightarrow ye^{2x} = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$\Rightarrow ye^{2x} = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$\Rightarrow y = \frac{1}{2}x - \frac{1}{4} + Ce^{-2x}$$

APPLY THE CONDITION $(0,1)$ TO FIND C

$$\Rightarrow 1 = 0 - \frac{1}{4} + C$$

$$\Rightarrow C = \frac{5}{4}$$

$$\therefore y = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x}$$

ALTERNATIVE SOLUTION BY SUBSTITUTION

$$V = x - 2y$$

$$\frac{dV}{dx} = 1 - 2\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x - 2y$$

$$\Rightarrow -2\frac{dy}{dx} = -2(x-2y)$$

$$\Rightarrow 1 - 2\frac{dy}{dx} = 1 - 2(x-2y)$$

$$\Rightarrow \frac{dV}{dx} = 1 - 2V$$

$$\Rightarrow \int \frac{1}{1-2V}\,dV = \int 1\,dx$$

$$\Rightarrow -\frac{1}{2}\ln|1-2V| = x + C$$

$$\Rightarrow \ln|1-2V| = -2x + D$$

$$\Rightarrow 1 - 2V = e^{-2x+D}$$

$$\Rightarrow 1 - 2V = Ae^{-2x}$$

$$\Rightarrow 1 - 2(x-2y) = Ae^{-2x}$$

$$\Rightarrow 1 - 2x + 4y = Ae^{-2x}$$

$$\Rightarrow 4y = 2x - 1 + Ae^{-2x}$$

$$\Rightarrow y = \frac{1}{2}x - \frac{1}{4} + Ee^{-2x}$$

AS ABOVE

IYGB - SYN F PAPER A - QUESTION 19

b) COLLECT SOME INFORMATION FIRST

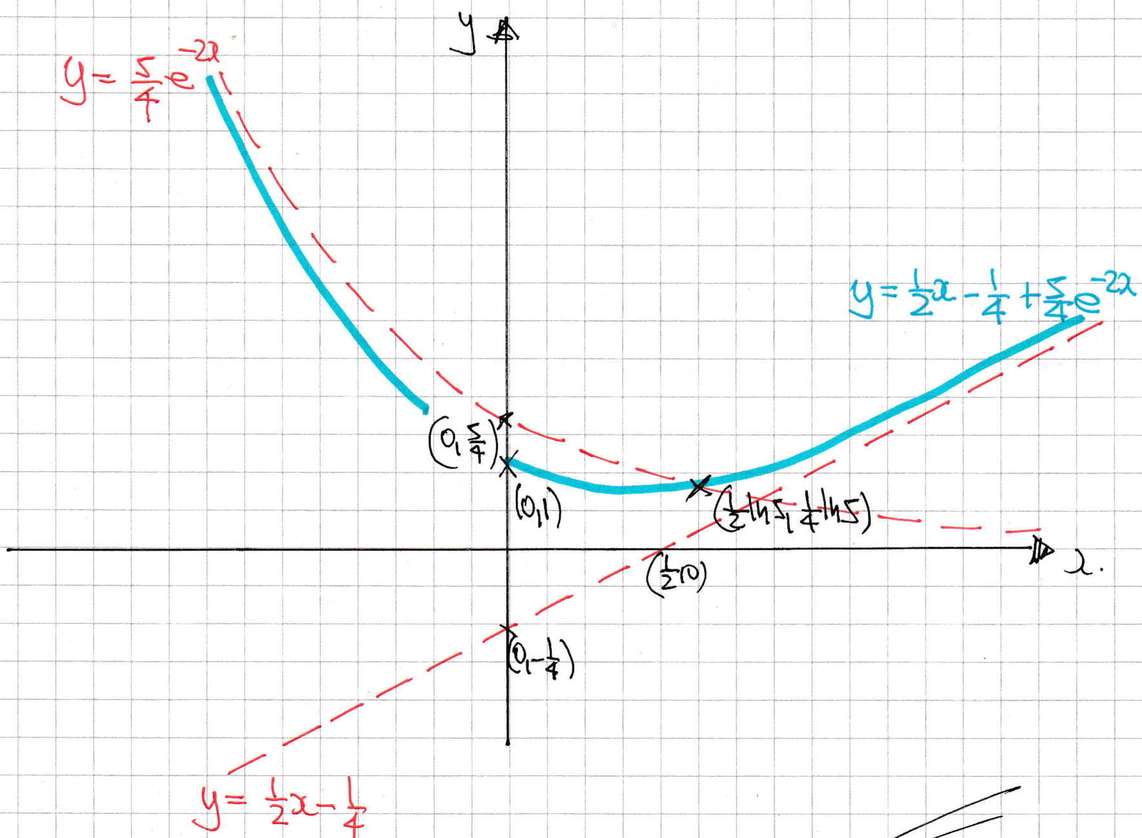
$$y = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x}$$
$$\frac{dy}{dx} = \frac{1}{2} - \frac{5}{2}e^{-2x}$$
$$0 = \frac{1}{2} - \frac{5}{2}e^{-2x}$$
$$5e^{-2x} = 1$$
$$e^{-2x} = \frac{1}{5}$$
$$e^{2x} = 5$$
$$x = \frac{1}{2}\ln 5$$

$$y = \frac{1}{2}\left(\frac{1}{2}\ln 5\right) - \frac{1}{4} + \frac{5}{4} \times \frac{1}{5}$$
$$y = \frac{1}{4}\ln 5 - \frac{1}{4} + \frac{1}{4}$$
$$y = \frac{1}{4}\ln 5$$

$\therefore$ STATIONARY AT $\left(\frac{1}{2}\ln 5, \frac{1}{4}\ln 5\right)$

Now as $x \to +\infty$, $y \sim \frac{1}{2}x - \frac{1}{4}$

As $x \to -\infty$, $y \sim \frac{5}{4}e^{-2x}$



IYGB - SYNF PAPER A - QUESTION 20

MANIPULATE THE INTEGRAL AS FOLLOWS

$$\int_0^{\frac{\pi}{4}} \frac{1}{1+3\sin^2 x}\,dx = \int \frac{1}{(\cos^2 x+\sin^2 x)+3\sin^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x + 4\sin^2 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1\sec^2 x}{\cos^2 x\sec^2 x + 4\sin^2 x\sec^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+4\tan^2 x}\,dx$$

NOW A SUBSTITUTION OR INSPECTION AS THIS IS AN ARCTAN DIFFERENTIAL

$$\frac{d}{dx}\left[\arctan(2\tan x)\right] = \frac{1}{1+4\tan^2 x} \times 2\sec^2 x = 2\left[\frac{\sec^2 x}{1+4\tan^2 x}\right]$$

THUS WE CAN EVALUATE

$$= \left[\frac{1}{2}\arctan(2\tan x)\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\arctan\left(2\tan\frac{\pi}{4}\right) - \cancel{\frac{1}{2}\arctan(2\times 0)}$$

$$= \frac{1}{2}\arctan 2$$

IYGB - FURTHER SYNOPTIC PAPER A - QUESTION 21

LET THE REQUIRED POSITIVE INTEGER BE N

- $N = 4Q + R$ WHERE $R = \cancel{0}, 1, 2, 3$
- $N = 12R + R$ $(Q = 3R)$
- $N = 13R$ WHERE $R = 1, 2, 3$

$\therefore$ $N = 13, 26, 39$

IYGB – SYNF PAPER A – QUESTION 22

a) $z_1 = 1 + \sqrt{3}i$

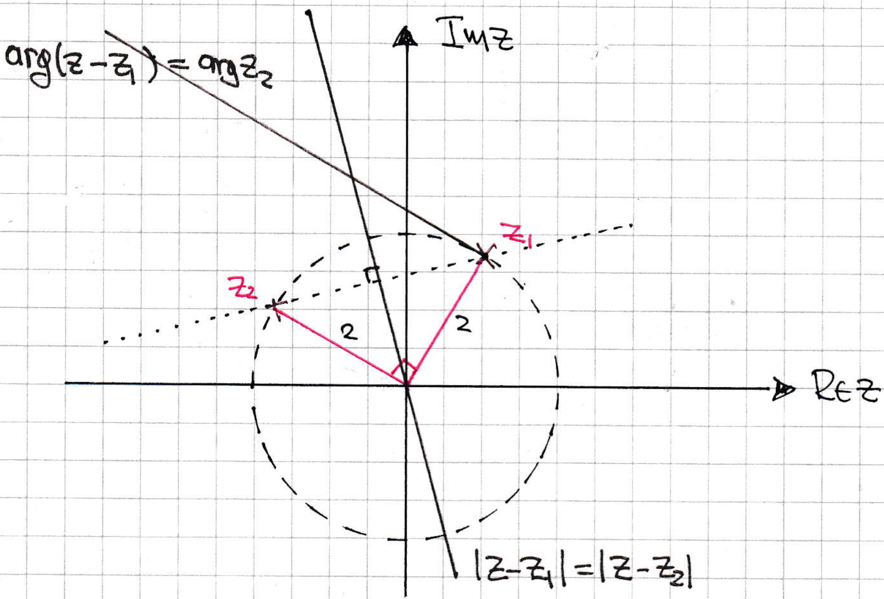
$|z_1| = \sqrt{1^2 + \sqrt{3}^2} = 2$

$\arg z_1 = \arctan \frac{\sqrt{3}}{1} = \frac{\pi}{3}$

$z_2 = iz_1 = -\sqrt{3} + i$

$|z_2| = \sqrt{(-\sqrt{3})^2 + 1^2} = 2$

$\arg z_2 = \arctan\left(\frac{-1}{\sqrt{3}}\right) + \pi = \frac{5\pi}{6}$



b) USING ABOVE DIAGRAM

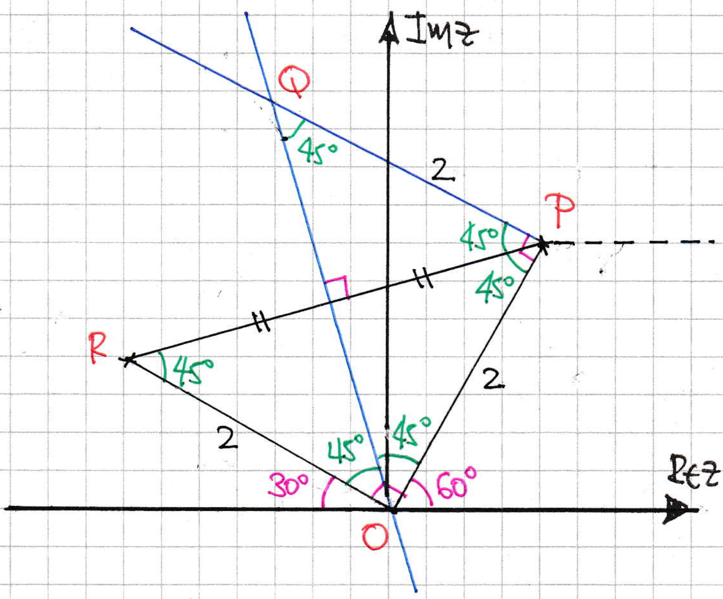
- $|z - z_1| = |z - z_2|$ IS THE PERPENDICULAR BISECTOR OF THE STRAIGHT LINE SEGMENT JOINING z_1 TO z_2

- $\arg(z - z_1) = \arg z_2$ IS A HALF LINE STARTING AT z_1, INCLINED TO THE POSITIVE HORIZONTAL BY $\frac{5\pi}{6}$ (IE THE ARGUMENT OF z_2)

P.T.O

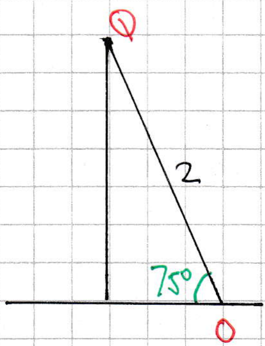
IYGB - SYNF PAPER A - QUESTION 22

c) LOOKING AT A NEW DIAGRAM WITH MORE DETAIL - USE DEGREES



NOTE PQ IS PARALLEL TO OR, SO THERE IS A RIGHT ANGLE AT $Q\hat{P}O$ AND HENCE $|PQ| = 2$ & OPQR IS A SQUARE & FURTHER TO THIS OQ IS EXACTLY $2\sqrt{2}$

FINALLY WE HAVE



- $x = -2\sqrt{2}\cos 75$

  $x = -2\sqrt{2}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$

  $x = 1-\sqrt{3}$

- $y = 2\sqrt{2}\sin 75$

  $y = 2\sqrt{2}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$

  $y = 1+\sqrt{3}$

$\therefore\ z_3 = (1-\sqrt{3}) + (1+\sqrt{3})i$

IYGB – FURTHER SYNOPTIC PAPER A – QUESTION 23

METHOD A

- LET A LINE THROUGH THE ORIGIN HAVE EQUATION $y = mx$, WHICH IS THEN MAPPED TO $Y = mX$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} X \\ mX \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ mx \end{bmatrix} = \begin{bmatrix} 3mx \\ 3x \end{bmatrix}$$

- HENCE WE OBTAIN THE EQUATIONS

$$\left.\begin{aligned} X &= 3mx \\ mX &= 3x \end{aligned}\right\} \Longrightarrow \text{DIVIDING THE EQUATIONS WE OBTAIN}$$

$$\frac{1}{m} = m$$

$$m^2 = 1$$

$$m = \pm 1$$

$\therefore$ THE REQUIRED LINES ARE $y = x$ & $y = -x$ //

METHOD B (BY EIGENVECTORS)

- FIND THE CHARACTERISTIC EQUATION OF $\underline{M}$

$$\begin{vmatrix} 0-\lambda & 3 \\ 3 & 0-\lambda \end{vmatrix} = 0 \Longrightarrow (-\lambda)^2 - 9 = 0$$

$$\Longrightarrow \lambda^2 - 9 = 0$$

$$\Longrightarrow (\lambda - 3)(\lambda + 3) = 0$$

$$\Longrightarrow \lambda = \begin{cases} -3 \\ 3 \end{cases}$$

IYGB – FURTHER SYNOPTIC PAPER A – QUESTION 23

- FINDING THE EIGENVECTORS AND HENCE THE LINES

IF $\lambda = 3$

$3y = 3x$

$3x = 3y$

$\therefore\ y = x$

IF $\lambda = -3$

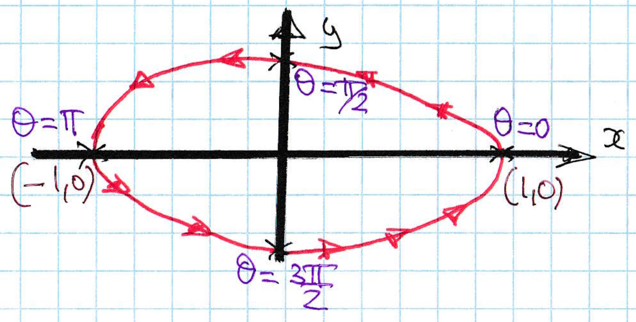
$3y = -3x$

$3x = -3y$

$\therefore\ y = -x$

IYGB − FURTHER SYNOPTIC PAPER A − QUESTION 24

FIRST DETERMINE THE "ORIENTATION/TRACING" OF THE CURVE, IN TERMS OF θ (BY INSPECTION)



USING THE SYMMETRY OF THE CURVE AND REVOLVING THE TOP HALF BY 2π

$$V = \pi\int_{x_1}^{x_2}[y(x)]^2\,dx = \pi\int_{\theta_1}^{\theta_2}[y(\theta)]^2\frac{dx}{d\theta}\,d\theta$$

$$= \pi\int_{\pi}^{0}\left(\sin\theta - \tfrac{1}{4}\sin 2\theta\right)^2(-\sin\theta)\,d\theta$$

$$= \pi\int_{0}^{\pi}\left[\sin\theta - \tfrac{1}{4}\times 2\sin\theta\cos\theta\right]^2\sin\theta\,d\theta$$

$$= \pi\int_{0}^{\pi}\sin^2\theta\left(1-\tfrac{1}{2}\cos\theta\right)^2\sin\theta\,d\theta$$

$$\therefore\ V = \pi\int_{0}^{\pi}\sin^3\theta\left(1-\tfrac{1}{2}\cos\theta\right)^2 d\theta$$

NOW LET $u=\cos\theta \implies \frac{du}{d\theta} = -\sin\theta$

$\implies d\theta = -\frac{du}{\sin\theta}$

θ	0	π
u	1	−1

$$\implies V = \pi\int_{1}^{-1}\sin^{\cancel{3}\,2}\theta\left(1-\tfrac{1}{2}u\right)^2\left(-\frac{du}{\cancel{\sin\theta}}\right)$$

$$\implies V = \pi\int_{-1}^{1}\sin^2\theta\left(1-\tfrac{1}{2}u\right)^2 du$$

$$\implies V = \pi\int_{-1}^{1}(1-\cos^2\theta)\left(1-\tfrac{1}{2}u\right)^2 du$$

$$\implies V = \pi\int_{-1}^{1}(1-u^2)\left(1-\tfrac{1}{2}u\right)^2 du$$

IYGB - FURTHER SYNOPTIC PAPER A - QUESTION 24

$$\Rightarrow V = \pi \int_{-1}^{1} (1-u^2)(1-u+\tfrac{1}{4}u^2)\, du$$

MULTIPLY OUT & THROW AWAY ODD PARTS AS THE DOMAIN IS SYMMETRICAL

$$\Rightarrow V = \pi \int_{-1}^{1} 1 \cancel{-u} + \tfrac{1}{4}u^2 - u^2 \cancel{+u^3} - \tfrac{1}{4}u^4 \, du$$

$$\Rightarrow V = \pi \int_{-1}^{1} 1 - \tfrac{3}{4}u^2 - \tfrac{1}{4}u^4 \, du$$

EVEN PARTS ×2

$$\Rightarrow V = \pi \int_{0}^{1} 2 - \tfrac{3}{2}u^2 - \tfrac{1}{2}u^4 \, du$$

$$\Rightarrow V = \pi \left[2u - \tfrac{1}{2}u^3 - \tfrac{1}{10}u^5 \right]_0^1$$

$$\Rightarrow V = \pi \left[2 - \tfrac{1}{2} - \tfrac{1}{10} \right]$$

$$\Rightarrow V = \tfrac{7}{5}\pi$$