IYGB GCE

Mathematics SYNF

Advanced Level

Synoptic Paper B

Difficulty Rating: 3.3375/0.5258

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Further Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 22 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

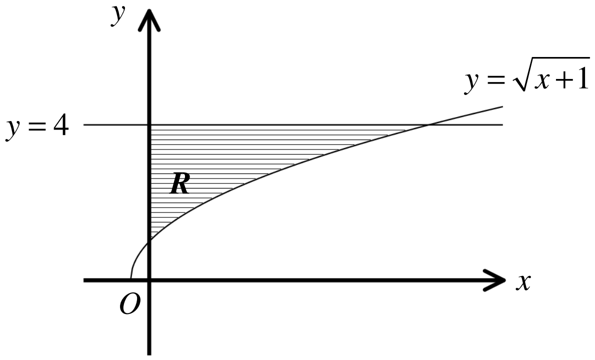
You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

Find the equation of the straight line which is common to the planes

$$x - 2y + 4z = 9 \quad \text{and} \quad 2x - 3y + z = 4. \qquad (7)$$

Question 2



The curve C has equation

$$y = \sqrt{x+1}, \ x > -1.$$

The region R is bounded by C, the y axis and the straight line with equation $y = 4$ is rotated through $360°$ about the y axis to form a solid of revolution.

Show that the volume of the solid is $\frac{828\pi}{5}$. (6)

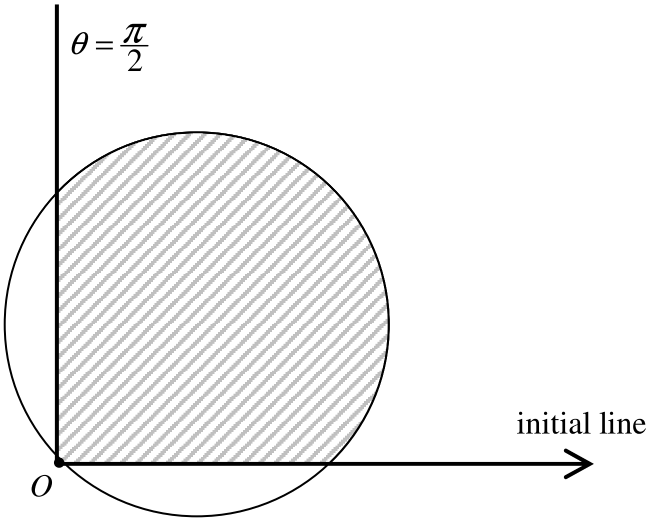
Question 3

$$\frac{x^2}{4} + y^2 = 1.$$

The ellipse with Cartesian equation above and a parabola with vertex at the origin share the same focal point.

Find the possible Cartesian equation for the parabola. (6)

Question 4



The figure above shows a circle with polar equation

$$r = 4(\cos\theta + \sin\theta) \quad 0 \le \theta < 2\pi .$$

a) Find the exact area of the shaded region bounded by the circle, the initial line and the half line $\theta = \frac{\pi}{2}$. **(5)**

b) Determine the Cartesian coordinates of the centre of the circle and the length of its radius. **(5)**

Question 5

Find a general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6e^x . \quad \textbf{(8)}$$

Question 6

The values of y for the curve C with equation $y = f(x)$ have been tabulated below.

x	–3	–1	1	3	5
y	6	12	18	25	a

The average value of $f(x)$ in the interval $(-3,5)$ is 17.

Use Simpson's rule with all the values from the table to find an estimate for the value of the constant a. **(7)**

Question 7

Show that if x is measured in radians, the general solution of

$$6\tan^2 x = 1 + 4\sin^2 x,$$

is given by

$$x = \frac{1}{6}\pi f(n),$$

where $f(n)$ is an integer function to be found. **(9)**

Question 8

By using elementary row and column operations, or otherwise, factorize the following determinant completely.

$$\begin{vmatrix} a & b & -c \\ b-c & a-c & a+b \\ -bc & -ca & ab \end{vmatrix}.$$

(6)

Question 9

With respect to a fixed origin O, the points A and B have position vectors given by

$$\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

a) Find a Cartesian equation of the plane that passes through O, A and B. **(3)**

A straight line has a vector equation

$$\left[\mathbf{r} - (4\mathbf{i} + \mathbf{j} + 6\mathbf{k})\right] \wedge (\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{0}.$$

b) Determine the coordinates of the point C, where C is the intersection between the straight line and the plane. **(4)**

Question 10

The roots of the quadratic equation

$$2x^2 - 3x + 5 = 0$$

are denoted by α and β.

Find the quadratic equation, with integer coefficients, whose roots are

$$3\alpha - \beta \quad \text{and} \quad 3\beta - \alpha.$$ **(7)**

Question 11

By using standard results, show that

$$\sum_{r=n+1}^{4n} (2r-1)^2 \equiv n\left(84n^2 - 1\right).$$ **(7)**

Question 12

The complex number z satisfies the following equation.

$$|z+8-16i| = |z|.$$

In a standard Argand diagram, the complex numbers represented by the points A and B lie on the real and imaginary axes, respectively.

Given further that A and B satisfy the above equation, determine an equation for the circle which passes though the points A, B and O, where O is the origin of the Argand diagram.

Give the answer in the form $|z-z_0| = r$, where $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}$. **(10)**

Question 13

The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix}.$$

a) Find $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$. **(2)**

b) Find a matrix $\mathbf{C}$, so that

$$(\mathbf{B}+\mathbf{C})^{-1} = \mathbf{A}.$$ **(5)**

c) Describe geometrically the transformation represented by $\mathbf{C}$. **(2)**

Question 14

By using the substitution $y = xv$, where $v = f(x)$, solve the differential equation

$$\frac{dy}{dx} = \frac{xy+y^2}{x^2}, \quad x>0$$

subject to the condition $y=-1$ at $x=1$. **(9)**

Question 15

a) Starting from the definitions of $\cosh x$ and $\sinh x$, in terms of exponentials, show that

$$\cos(\mathrm{i}\varphi) \equiv \cosh(\varphi) \quad \text{and} \quad \sin(\mathrm{i}\varphi) \equiv \mathrm{i}\sinh(\varphi). \qquad (6)$$

b) Use the results of part **(a)** to deduce

$$\operatorname{sech}^2\varphi + \tanh^2\varphi \equiv 1. \qquad (4)$$

c) Hence find, in exact logarithmic form, the solutions of the following equation.

$$10\operatorname{sech} y = 5 + 3\tanh^2 y. \qquad (6)$$

Question 16

$$I(a,x) \equiv \int \sqrt{x^2+a^2}\, dx, \quad x \in \mathbb{R}, \quad a \in \mathbb{R}, \quad x > 0.$$

a) Use a suitable hyperbolic substitution to show that

$$I(a,x) = \frac{1}{2}a^2\left[\operatorname{arsinh}\left(\frac{x}{a}\right) + \frac{x\sqrt{x^2+a^2}}{a^2}\right] + \text{constant}. \qquad (7)$$

b) Hence find in exact form the length of the curve with equation

$$y = \frac{1}{4}x^2,$$

from the origin O to the point with coordinates $\left(1, \frac{1}{4}\right)$. **(5)**

Question 17

The complex number is defined as $z = \cos\theta + \mathrm{i}\sin\theta$, $-\pi < \theta \le \pi$.

a) Show clearly that …

i. … $z^n + \dfrac{1}{z^n} = 2\cos\theta$. **(3)**

ii. … $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin\theta$. **(2)**

iii. … $8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$. **(5)**

b) Hence solve the equation

$$8\sin^4\theta + 5\cos 2\theta = 3, \quad -\pi < \theta \le \pi.$$

(6)

Question 18

A curve C has equation

$$y = \frac{1}{x^3 - 9x^2 + 24x}, \quad x \in \mathbb{R}, \ x \ne 0.$$

Sketch the graph of C.

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of C meets the coordinate axes.
- the coordinates of any stationary points.
- the equations of any asymptotes.

You must further label any non stationary turning points, without explicitly giving their coordinates.

(10)

Question 19

In the following question A, B and C are positive odd integers.

Show, using a clear method, that …

a) … $A^2 + B^2 + C^2 + 5$ is a multiple of 8. **(3)**

b) … $A^2\left(A^2 + 6\right) - 7$ is divisible by 128. **(4)**

c) … $A^4 - B^4$ is a multiple of 16. **(4)**

Question 20

a) If $p \in (0,\infty)$, show that

$$\lim_{x \to 0^+}\left[x^p \ln x\right] = 0, \quad x \in (0,\infty). \qquad \mathbf{(4)}$$

b) Hence find a simplified expression for

$$\int_0^1 x^n \ln x \, dx, \ n \in \mathbb{N}. \qquad \mathbf{(5)}$$

Question 21

Find the value of the following definite integral.

$$\int_0^{\frac{1}{2}} \frac{12x - 1}{\left(6x^2 - x - 1\right)\left(6x^2 - x - 5\right) + 10} \, dx .$$

Give the answer in the form $\arctan\left(\frac{1}{n}\right)$, where n is a positive integer. **(10)**

Question 22

A linear transformation T, acting in the x-y plane, consists of …

- … an anticlockwise rotation about the origin by $\frac{\pi}{2}$

followed by ...

- … a translation by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Determine the coordinates of the invariant point under T. **(8)**