IYGB - SYNF PAPER B - QUESTION 1

FIRST APPROACH (BY ROW REDUCING)

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 9 \\ 2 & -3 & 1 & 4 \end{array}\right] \quad r_{12}(-2) \quad \left[\begin{array}{cccc} 1 & -2 & 4 & 9 \\ 0 & 1 & -7 & -14 \end{array}\right] \quad r_{21}(2)$$

$$\left[\begin{array}{cccc} 1 & 0 & -10 & -19 \\ 0 & 1 & -7 & -14 \end{array}\right] \qquad \text{IE} \quad \begin{array}{l} x - 10z = -19 \\ y - 7z = -14 \end{array}$$

$$\begin{array}{l} x = -19 + 10z \\ y = -14 + 7z \\ z = 0 + z \end{array} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -19 \\ -14 \\ 0 \end{pmatrix} + t\begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}$$

TIDY A BIT TO MAKE THE NUMBERS SMALLER A BIT.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -19 \\ -14 \\ 0 \end{pmatrix} + (\lambda+2)\begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -19 \\ -14 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix} + \begin{pmatrix} 20 \\ 14 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}$$

SECOND APPROACH – FIND THE DIRECTION BY CROSSING THE NORMALS

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 4 \\ 2 & -3 & 1 \end{vmatrix} = (-2+12,\ 8-1,\ -3+4) = (10,7,1)$$

NOW LET SAY $x=1$ IN THE EQUATIONS.

$$\left.\begin{array}{l} 1-2y+4z=9 \\ 2-3y+z=4 \end{array}\right\} \Longrightarrow \left.\begin{array}{l} -2y+4z=8 \\ -3y+z=2 \end{array}\right\} \Rightarrow z = 3y+2$$

SUB INTO THE OTHER

$$\Rightarrow \ -2y+4(3y+2)=8$$

IYGB - SYNF PAPER B - QUESTION 1

$$\Rightarrow -2y + 10y + 8 = 8$$

$$\Rightarrow 8y = 0$$

$$\Rightarrow y = 0$$

AND SINCE $z = 3y+2$, $z = 2$

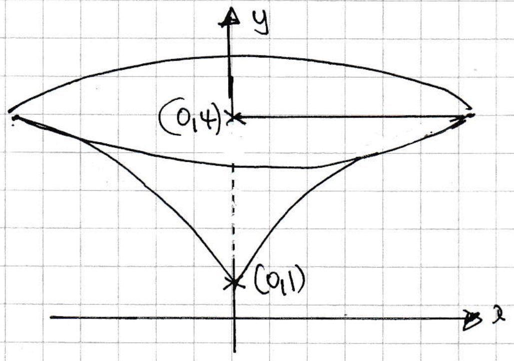
USING THE COMMON POINT $(1,0,2)$ AND DIRECTION $(10,7,1)$

$$\underline{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}$$

// AS BEFORE

IYGB - FURTHER SYNOPTIC PAPER B - QUESTION 2

FIRSTLY BY INSPECTION THE CURVE MEETS THE y AXIS AT $(0,1)$



$y = \sqrt{x+1}$

$y^2 = x+1$

$x = y^2 - 1$

$x^2 = (y^2-1)^2$

$x^2 = y^4 - 2y^2 + 1$

SETTING UP A VOLUME INTEGRAL, ABOUT THE y AXIS

$$\text{VOLUME} = \pi\int_{y_1}^{y_2} (x(y))^2 \, dy = \pi\int_1^4 (y^4 - 2y^2 + 1) \, dy$$

$$= \pi\left[\frac{1}{5}y^5 - \frac{2}{3}y^3 + y\right]_1^4 = \pi\left(\frac{1024}{5} - \frac{128}{3} + 4\right) - \pi\left(\frac{1}{5} - \frac{2}{3} + 1\right)$$

$$= \frac{2492}{15}\pi - \frac{8}{15}\pi$$

$$= \frac{828}{5}\pi$$

// AS REQUIRED

IYGB - SYNF PAPER B - QUESTION 3

LOOK FOR THE ECCENTRICITY OF THE ELLIPSE

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$\underset{a^2=4}{\uparrow} \qquad \underset{b^2=1}{\uparrow}$$

$$\Rightarrow b^2 = a^2(1-e^2)$$
$$\Rightarrow 1 = 4(1-e^2)$$
$$\Rightarrow \frac{1}{4} = 1-e^2$$
$$\Rightarrow e^2 = \frac{3}{4}$$
$$\Rightarrow e = +\frac{\sqrt{3}}{2}$$

THE ELLIPSE HAS FOCI AT $(\pm ae, 0)$, I.E. $\left(\pm\frac{\sqrt{3}}{2}\times 2, 0\right)$, $(\pm\sqrt{3}, 0)$

THE PARABOLA HAS EQUATION $y^2 = 4ax$

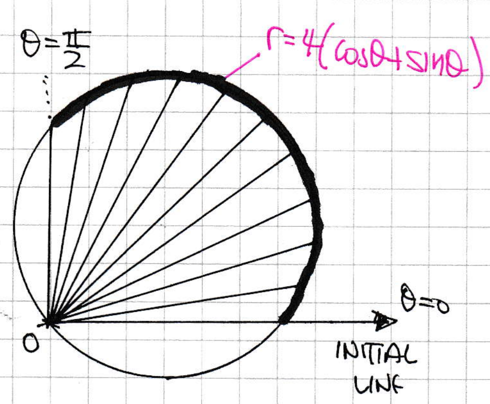
$$\underset{\pm\sqrt{3}}{\uparrow}$$

$$\therefore \quad y^2 = 4\sqrt{3}x \quad \text{OR} \quad y^2 = -4\sqrt{3}x$$

$$y^2 = \sqrt{48}x \quad \text{OR} \quad y^2 = -\sqrt{48}x$$

IYGB - SYNF PAPER B - QUESTION 4

a) USING THE STANDARD FORMULA



$$\Rightarrow \text{AREA} = \frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2 \, d\theta$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\int_{0}^{\pi/2} 16(\cos\theta+\sin\theta)^2 \, d\theta$$

$$\Rightarrow \text{AREA} = \frac{1}{2}\int_{0}^{\pi/2} 16(\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta) \, d\theta$$

$$\Rightarrow \text{AREA} = \int_{0}^{\pi/2} 8(1+2\cos\theta\sin\theta) \, d\theta$$

$$\Rightarrow \text{AREA} = 8\Big[\theta+\sin^2\theta\Big]_0^{\pi/2}$$

$$\Rightarrow \text{AREA} = 8\Big[\left(\tfrac{\pi}{2}+1\right)-0\Big]$$

$$\Rightarrow \text{AREA} = 4\pi+8$$

b) PROBABLY EASIER TO WORK IN CARTESIAN

$x=r\cos\theta$
$y=r\sin\theta$
$x^2+y^2=r$

$$\Rightarrow r = 4(\cos\theta+\sin\theta)$$

$$\Rightarrow r = 4\left(\frac{x}{r}+\frac{y}{r}\right)$$

$$\Rightarrow r^2 = 4x+4y$$

$$\Rightarrow x^2+y^2 = 4x+4y$$

$$\Rightarrow x^2-4x+y^2-4y=0$$

$$\Rightarrow (x-2)^2-4+(y-2)^2-4=0$$

$$\Rightarrow (x-2)^2+(y-2)^2=8$$

$\therefore$ CENTRE AT $(2,2)$

RADIUS $2\sqrt{8}$

IYGB - SYNF PAPER B - QUESTION 5

START WITH AN AUXILIARY EQUATION

$$\lambda^2 + \lambda - 2 = 0$$
$$(\lambda - 1)(\lambda + 2) = 0$$
$$\lambda = \begin{cases} 1 \\ -2 \end{cases}$$

COMPLEMENTARY FUNCTION

$$y = Ae^x + Be^{-2x}$$

NOW FOR PARTICULAR INTEGRAL WE TRY $y = Pxe^x$ AS e^x IS ALREADY PART OF THE COMPLEMENTARY FUNCTION

$$y = Pxe^x$$
$$\frac{dy}{dx} = Pe^x + Pxe^x = P(1+x)e^x$$
$$\frac{d^2y}{dx^2} = Pe^x + Pe^x + Pxe^x = 2Pe^x + Pxe^x = Pe^x(2+x)$$

SUBSTITUTE INTO THE O.D.E.

$$P\cancel{e^x}(2+x) + P(1+x)\cancel{e^x} - 2Px\cancel{e^x} \equiv 6\cancel{e^x}$$
$$P[2+x+1+x-2x] \equiv 6$$
$$3P = 6$$
$$P = 2$$

∴ GENERAL SOLUTION IS

$$y = Ae^x + Be^{-2x} + 2xe^x$$

IYGB - SYN F PAPER B - QUESTION 6

x	-3	-1	1	3	5
$y=f(x)$	6	12	18	25	a
	FIRST	ODD	EVEN	ODD	LAST

USING SIMPSONS RULE

$$\int_{-3}^{5} f(x)\,dx \simeq \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODDS} + 2\times\text{EVENS}\right]$$

$$\approx \frac{2}{3}\left[6 + a + 4(12+25) + 2\times 18\right]$$

$$\simeq \frac{2}{3}\left[a + 190\right]$$

$$\approx \frac{2}{3}a + \frac{380}{3}$$

NOW THE AVERAGE VALUE OF THE FUNCTION IS 17

$$\Rightarrow \frac{\int_{-3}^{5} f(x)\,dx}{5-(-3)} = 17$$

$$\Rightarrow \frac{\frac{2}{3}a + \frac{380}{3}}{8} = 17$$

$$\Rightarrow \frac{2}{3}a + \frac{380}{3} = 136$$

$$\Rightarrow 2a + 380 = 408$$

$$\Rightarrow 2a = 28$$

$$\Rightarrow a = 14$$

//

IYGB – SYNF PAPER B – QUESTION 7

SWITCHING INTO SINES AND COSINES

$$6\tan^2 x = 1 + 4\sin^2 x$$

$$\frac{6\sin^2 x}{\cos^2 x} = 1 + 4\sin^2 x$$

$$6\sin^2 x = \cos^2 x(1 + 4\sin^2 x)$$

$$6\sin^2 x = (1 - \sin^2 x)(1 + 4\sin^2 x)$$

$$6\sin^2 x = 1 + 4\sin^2 x - \sin^2 x - 4\sin^4 x$$

$$4\sin^4 x + 3\sin x - 1 = 0$$

FACTORIZING AS A QUADRATIC IN $\sin^2 x$

$$(4\sin^2 x - 1)(\sin^2 x + 1) = 0$$

$$\sin^2 x = \begin{cases} 1/4 \\ \cancel{-1} \end{cases}$$

$$\sin x = \begin{cases} 1/2 \\ -1/2 \end{cases}$$

SETTING UP THE GENERAL SOLUTION

$$\arcsin\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

$$x = n\pi + (-1)^n \frac{\pi}{6}$$

$$x = \frac{\pi}{6}\left[6n + (-1)^n\right]$$

$$\arcsin\left(-\tfrac{1}{2}\right) = -\frac{\pi}{6}$$

$$x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$$

$$x = n\pi - (-1)^n \frac{\pi}{6}$$

$$x = \frac{\pi}{6}\left[6n - (-1)^n\right]$$

$$\therefore\ x = \frac{\pi}{6}\left[6n \pm (-1)^n\right]$$

IE $f(n) = 6n \pm (-1)^n$

IYGB - SYNF PAPER B - QUESTION 8

$$\begin{vmatrix} a & b & -c \\ b-c & a-c & a+b \\ -bc & -ca & ab \end{vmatrix} \overset{r_{12}(1)}{=} \begin{vmatrix} a & b & -c \\ a+b-c & a+b-c & a+b-c \\ -bc & -ca & ab \end{vmatrix}$$

$$=(a+b-c)\begin{vmatrix} a & b & -c \\ 1 & 1 & 1 \\ -bc & -ca & ab \end{vmatrix} \underset{C_{13}(-1)}{\overset{C_{12}(-1)}{=}} (a+b+c)\begin{vmatrix} a & b-a & -c-a \\ 1 & 0 & 0 \\ -bc & bc-ca & ab+bc \end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix} a & b-a & -(c+a) \\ 1 & 0 & 0 \\ -bc & c(b-a) & b(a+c) \end{vmatrix} = (a+b+c)(b-a)(a+c)\begin{vmatrix} a & 1 & -1 \\ 1 & 0 & 0 \\ -bc & c & b \end{vmatrix}$$

EXPAND BY THE 2^{ND} ROW

$$= (a+b+c)(b-a)(a+c)\left[-1 \begin{vmatrix} 1 & -1 \\ c & b \end{vmatrix} \right]$$

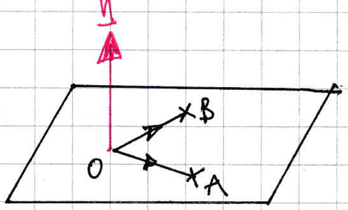
$$= (a+b+c)(b-a)(a+c)(-c-b)$$

$$= (a+b+c)(a-b)(b+c)(c+a)$$

IYGB - SYNF PAPER B - QUESTION 9

a) OBTAIN A NORMAL

$$\vec{OA} \wedge \vec{OB} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = (-1, 7, 5)$$



USING THE ORIGIN $(0,0,0)$

$$-x + 7y + 5z = 0$$

$$x - 7y - 5z = 0$$

b) REWRITE THE EQUATION OF THE LINE IN PARAMETRIC

$$\underline{r} = (4,1,6) + \lambda(1,1,1)$$

$$\underline{r} = (\lambda+4, \lambda+1, \lambda+6)$$

SOLVING SIMULTANEOUSLY

- $x = \lambda+4$
- $y = \lambda+1$
- $z = \lambda+6$

& $x - 7y - 5z = 0$

$$\Rightarrow (\lambda+4) - 7(\lambda+1) - 5(\lambda+6) = 0$$

$$\Rightarrow \lambda + 4 - 7\lambda - 7 - 5\lambda - 30 = 0$$

$$\Rightarrow -11\lambda = 33$$

$$\Rightarrow \lambda = -3$$

$$\therefore C(1,-2,3)$$

IYGB - SYNF PAPER B - QUESTION 10

LOOKING AT THE QUADRATIC

$2x^2-3x+5=0$

- $\alpha+\beta=-\frac{b}{a}=-\frac{-3}{2}=\frac{3}{2}$
- $\alpha\beta=\frac{c}{a}=\frac{5}{2}$

LET $A=3\alpha-\beta$ & $B=3\beta-\alpha$

SUM OF ROOTS

$A+B=(3\alpha-\beta)+(3\beta-\alpha)=2\alpha+2\beta=2(\alpha+\beta)=2\times\frac{3}{2}=3$

PRODUCT OF ROOTS

$AB=(3\alpha-\beta)(3\beta-\alpha)=9\alpha\beta-3\alpha^2-3\beta^2+\alpha\beta=10\alpha\beta-3(\alpha^2+\beta^2)$

$=10\alpha\beta-3\left[(\alpha+\beta)^2-2\alpha\beta\right]=10\times\frac{5}{2}-3\left[\left(\frac{3}{2}\right)^2-2\times\frac{5}{2}\right]$

$=25+\frac{33}{4}$

$=\frac{133}{4}$

FINALLY WE HAVE

$x^2-(A+B)x+(AB)=0$

$x^2-3x+\frac{133}{4}=0$

$4x^2-12x+133=0$

IYGB - FURTHER SYNOPTIC PAPER B - QUESTION 11

PROCEED AS FOLLOWS

$$f(n)=\sum_{r=1}^{n}(2r-1)^2=\sum_{r=1}^{n}\left[4r^2-4r+1\right]=4\sum_{r=1}^{n}r^2-4\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$$

USING STANDARD RESULTS

$$f(n)=4\times\tfrac{1}{6}n(n+1)(2n+1)-4\times\tfrac{1}{2}n(n+1)+n$$

$$f(n)=\tfrac{2}{3}n(n+1)(2n+1)-2n(n+1)+n$$

$$f(n)=\tfrac{1}{3}n\left[2(n+1)(2n+1)-6(n+1)+3\right]$$

$$f(n)=\tfrac{1}{3}n\left[4n^2+\cancel{6n}+2-\cancel{6n}-6+3\right]$$

$$f(n)=\tfrac{1}{3}n\left(4n^2-1\right)$$

NOW WE HAVE

$$\sum_{r=n+1}^{4n}(2r-1)^2=\sum_{r=1}^{4n}(2r-1)^2-\sum_{r=1}^{n}(2r-1)^2$$

$$=f(4n)-f(n)$$

$$=\tfrac{1}{3}(4n)\left[4(4n)^2-1\right]-\tfrac{1}{3}n\left[4n^2-1\right]$$

$$=\tfrac{4}{3}n\left(64n^2-1\right)-\tfrac{1}{3}n\left(4n^2-1\right)$$

$$=\tfrac{1}{3}n\left[256n^2-4-4n^2+1\right]$$

$$=\tfrac{1}{3}n\left(252n^2-3\right)$$

$$=n\left(84n^2-1\right)$$

AS REQUIRED

IYGB - FURTHER SYNOPTIC PAPER B - QUESTION 12

START BY OBTAINING A CARTESIAN EQUATION OF THE LINE

$$\Rightarrow |z+8-16i| = |z|$$

$$\Rightarrow |x+iy+8-16i| = |x+iy|$$

$$\Rightarrow |(x+8)+i(y-16)| = |x+iy|$$

$$\Rightarrow \sqrt{(x+8)^2+(y-16)^2} = \sqrt{x^2+y^2}$$

$$\Rightarrow (x+8)^2+(y-16)^2 = x^2+y^2$$

$$\Rightarrow \cancel{x^2}+16x+64+\cancel{y^2}-32y+256 = \cancel{x^2}+\cancel{y^2}$$

$$\Rightarrow 16x-32y = -320$$

$$\Rightarrow 32y-16x = 320$$

$$\Rightarrow \underline{2y-x = 20}$$

OBTAIN THE AXES INTERCEPTS & HENCE THE MIDPOINT OF AB

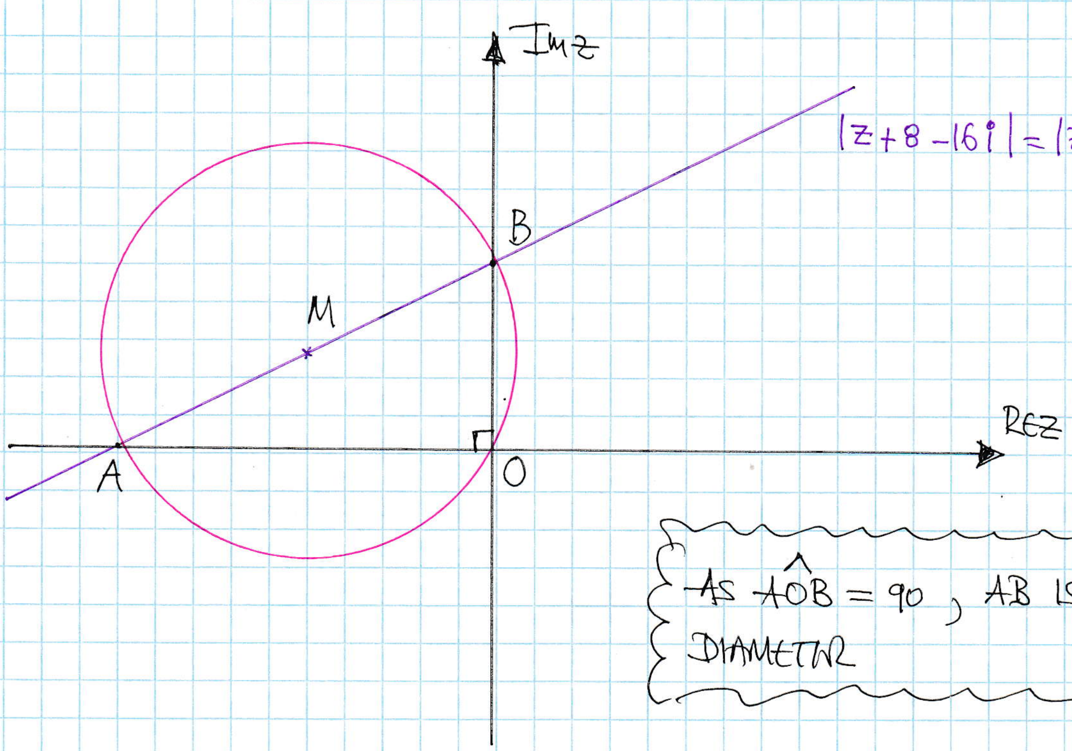
- $x=0 \quad 2y=20$
  $y=10$
  $\therefore \underline{B(0,10)}$
- $y=0 \quad -x=20$
  $x=-20$
  $\therefore \underline{A(-20,0)}$
- MIDPOINT OF AB $\therefore \underline{M(-10,5)}$

NEXT THE DISTANCE $|AB|$ — OR $|AM|$ OR $|BM|$

$$|AB| = \sqrt{(-20)^2+10^2} = \sqrt{400+100} = \underline{10\sqrt{5}}$$

IYGB - FURTHER SYNOPTIC PAPER B - QUESTION 12

LOOKING AT THE DIAGRAM BELOW



AS $A\hat{O}B = 90$, AB IS A DIAMETER

$\therefore$ CENTRE AT $M(-10,5)$

RADIUS $r = \frac{1}{2}|AB| = 5\sqrt{5}$

$\therefore$ $|z-(-10+5i)| = 5\sqrt{5}$

$\underline{\underline{|z+10-5i| = 5\sqrt{5}}}$

IYGB – SYNF PAPER B – QUESTION 13

a) STANDARD METHOD FOR INVERTING 2x2 MATRICES

$$|\underline{A}| = (4\times1)-(3\times1) = 4-3 = 1$$

$$\therefore \underline{A}^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}$$

b) PROCEED AS FOLLOWS

$$\Rightarrow (\underline{B}+\underline{C})^{-1} = \underline{A}$$

$$\Rightarrow (\underline{B}+\underline{C})(\underline{B}+\underline{C})^{-1} = (\underline{B}+\underline{C})\underline{A}$$

$$\Rightarrow \underline{I} = \underline{B}\underline{A} + \underline{C}\underline{A}$$

$$\Rightarrow \underline{C}\underline{A} = \underline{I} - \underline{B}\underline{A}$$

$$\Rightarrow \underline{C}\underline{A}\underline{A}^{-1} = (\underline{I} - \underline{B}\underline{A})\underline{A}^{-1}$$

$$\Rightarrow \underline{C}\underline{I} = \underline{I}\underline{A}^{-1} - \underline{B}\underline{A}\underline{A}^{-1}$$

$$\Rightarrow \underline{C} = \underline{A}^{-1} - \underline{B}\underline{I}$$

$$\Rightarrow \underline{C} = \underline{A}^{-1} - \underline{B}$$

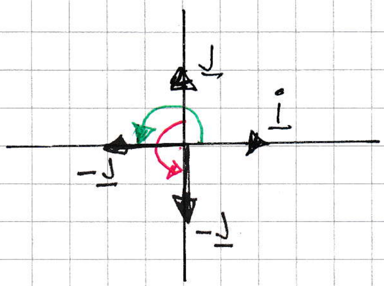
$$\Rightarrow \underline{C} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix}$$

$$\Rightarrow \underline{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

c) LOOKING AT BASE VECTORS $\underline{i}$ & $\underline{j}$

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\underline{i}$$

$$\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\underline{j}$$



$\therefore$ ROTATION ABOUT O, BY 180°

IYGB – SYNF PAPER B – QUESTION 14

AS THIS IS A FIRST ORDER HOMOGENEOUS EQUATION, USE $y = xV(x)$

$$\frac{dy}{dx} = 1 \times V(x) + x\frac{dV(x)}{dx} = V + x\frac{dv}{dx}$$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow \frac{dy}{dx} = \frac{xy + y^2}{x^2}$$

$$\Rightarrow V + x\frac{dv}{dx} = \frac{x(xv) + (xv)^2}{x^2}$$

$$\Rightarrow V + x\frac{dv}{dx} = \frac{x^2v + x^2v^2}{x^2}$$

$$\Rightarrow V + x\frac{dv}{dx} = V + v^2$$

$$\Rightarrow x\frac{dv}{dx} = v^2$$

$$\Rightarrow \frac{1}{v^2}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\frac{1}{v} = \ln|x| + C$$

$$\Rightarrow -\frac{1}{y/x} = \ln|x| + C$$

$$\Rightarrow -\frac{x}{y} = \ln|x| + C$$

$$\Rightarrow y = -\frac{x}{\ln|x| + C}$$

$$\Rightarrow y = -\frac{x}{\ln x + C}, \quad x > 0$$

APPLY BOUNDARY CONDITION $(1, -1)$

$$\Rightarrow -1 = -\frac{1}{\cancel{\ln 1} + C}$$

$$\Rightarrow C = 1$$

FINALLY WE HAVE

$$y = -\frac{x}{\ln x + 1}$$

$$y = -\frac{x}{1 + \ln x}$$

IYGB - FURTHER SYNOPTIC PAPER B - QUESTION 15

a) STARTING BY THE DEFINITIONS OF $\sinh x$ & $\cosh x$ IN EXPONENTIALS

- $\cosh x \equiv \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

LET $x = i\theta$

$$\Rightarrow \cosh(i\theta) = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$$

$$\Rightarrow \cosh(i\theta) = \frac{1}{2}\cos\theta + \cancel{\frac{1}{2}i\sin\theta} + \frac{1}{2}\cos\theta - \cancel{\frac{1}{2}i\sin\theta}$$ (BY EULER'S FORMULA)

$$\Rightarrow \cosh(i\theta) = \cos\theta$$

NOW LET $\theta = i\phi$

$$\Rightarrow \cosh(i(i\phi)) = \cos(i\phi)$$

$$\Rightarrow \cosh(-\phi) = \cos(i\phi)$$

$$\Rightarrow \cos(i\phi) \equiv \cosh\phi$$ (AS $\cosh\phi$ IS EVEN)

IN A SIMILAR FASHION

- $\sinh x \equiv \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$

LET $x = i\theta$

$$\Rightarrow \sinh(i\theta) = \frac{1}{2}e^{i\theta} - \frac{1}{2}e^{-i\theta}$$

$$\Rightarrow \sinh(i\theta) = \left(\cancel{\frac{1}{2}\cos\theta} + \frac{1}{2}i\sin\theta\right) - \left(\cancel{\frac{1}{2}\cos\theta} - \frac{1}{2}i\sin\theta\right)$$ (BY EULER'S FORMULA)

$$\Rightarrow \sinh(i\theta) = i\sin\theta$$

NOW LET $\theta = i\phi$

$$\Rightarrow \sinh(i(i\phi)) = i\sin(i\phi)$$

$$\Rightarrow \sinh(-\phi) = i\sin(i\phi)$$

$$\Rightarrow -\sinh\phi = i\sin(i\phi)$$ (AS $\sinh\phi$ IS ODD

$$\Rightarrow \sin(i\phi) = -\frac{1}{i}\sinh\phi$$

$$\Rightarrow \sin(i\phi) = i\sinh\phi$$ AS REQUIRED

IYGB - FURTHER SYNOPTIC PAPER A - QUESTION 15

b) STARTING WITH THE STANDARD IDENTITY $\cos^2 x + \sinh^2 x \equiv 1$

$$\Rightarrow \cos^2(i\phi) + \sin^2(i\phi) = 1$$

$$\Rightarrow \cos(i\phi)\cos(i\phi) + \sin(i\phi)\sin(i\phi) = 1$$

$$\Rightarrow \cosh\phi\cosh\phi + (i\sinh\phi)(i\sinh\phi) = 1$$

$$\Rightarrow \cosh^2\phi - \sinh^2\phi = 1$$

$$\Rightarrow \frac{\cosh^2\phi}{\cosh^2\phi} - \frac{\sinh^2\phi}{\cosh^2\phi} = \frac{1}{\cosh^2\phi}$$

$$\Rightarrow 1 - \tanh^2\phi = \operatorname{sech}^2\phi$$

$$\Rightarrow \operatorname{sech}^2\phi + \tanh^2\phi = 1$$

// AS REQUIRED

c) FINALLY USING PART (b)

$$\Rightarrow 10\operatorname{sech}y = 5 + 3\tanh^2 y$$

$$\Rightarrow 10\operatorname{sech}y = 5 + 3(1 - \operatorname{sech}^2 y)$$

$$\Rightarrow 10\operatorname{sech}y = 8 - 3\operatorname{sech}^2 y$$

$$\Rightarrow 3\operatorname{sech}^2 y + 10\operatorname{sech}y - 8 = 0$$

$$\Rightarrow (3\operatorname{sech}y - 2)(\operatorname{sech}y + 4) = 0$$

$$\Rightarrow \operatorname{sech}y = \begin{cases} -4 \\ \frac{2}{3} \end{cases}$$

$$\Rightarrow \cosh y = \begin{cases} \cancel{-\frac{1}{4}} \\ \frac{3}{2} \end{cases} \qquad (\cosh y \geqslant 1)$$

$$\Rightarrow y = \pm \operatorname{arccosh}\frac{3}{2}$$

$$\Rightarrow y = \pm \ln\left[\frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 - 1}\right]$$

$$\Rightarrow y = \pm \ln\left[\frac{3}{2} + \frac{\sqrt{5}}{2}\right]$$

$$\Rightarrow y = \pm \ln\left(\frac{3 + \sqrt{5}}{2}\right)$$

//

(FURTHER) IYGB – SYNOPTIC PAPER B – QUESTION 16

a) USING A HYPERBOLIC SUBSTITUTION

- $x = a\sinh\theta$
- $dx = a\cosh\theta$
- $\theta = \text{arsinh}\frac{x}{a}$

TRANSFORMING WE OBTAIN

$$I = \int \sqrt{x^2+a^2}\,dx = \int \sqrt{a^2\sinh^2\theta + a^2}\,(a\cosh\theta\,d\theta)$$

$$= \int \sqrt{a^2(\sinh^2\theta+1)}\,(a\cosh\theta)\,d\theta = \int \sqrt{a^2\cosh^2\theta}\,(a\cosh\theta\,d\theta)$$

$$= \int a^2\cosh^2\theta\,d\theta = \int a^2\left(\tfrac{1}{2}+\tfrac{1}{2}\cosh2\theta\right)d\theta = a^2\int \tfrac{1}{2}+\tfrac{1}{2}\cosh2\theta\,d\theta$$

$$= a^2\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sinh2\theta + C\right] = a^2\left[\tfrac{1}{2}\theta + \tfrac{1}{2}\sinh\theta\cosh\theta\right] + C$$

$$= a^2\left[\tfrac{1}{2}\text{arsinh}\left(\frac{x}{a}\right) + \tfrac{1}{2}\left(\frac{x}{a}\right)\underbrace{\sqrt{1+\left(\frac{x}{a}\right)^2}}_{\cosh\theta}\right] + C$$

$$= \tfrac{1}{2}a^2\left[\text{arsinh}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{\frac{a^2+x^2}{a^2}}\right] + C$$

$$= \tfrac{1}{2}a^2\left[\text{arsinh}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2+x^2}}{a^2}\right] + C$$

b) SETTING UP AN ARCLENGTH INTEGRAL

$$s = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$s = \int_0^1 \sqrt{1+\tfrac{1}{4}x^2}\,dx$$

$$y = \tfrac{1}{4}x^2$$

$$\frac{dy}{dx} = \tfrac{1}{2}x$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\tfrac{1}{4}x^2$$

IYGB - FURTHER SYNOPTIC PAPER B - QUESTION 16

$$\Rightarrow \quad s = \int_0^1 \frac{1}{2}\sqrt{4+x^2}\,dx$$

USING PART (a) WITH $a=2$

$$s = \frac{1}{4}\times 2^2\left[\operatorname{arsinh}\frac{x}{2} + \frac{x\sqrt{x^2+4}}{2^2}\right]_0^1$$

$$s = \left[\operatorname{arsinh}\frac{x}{2} + \frac{1}{4}x\sqrt{x^2+4}\right]_0^1$$

$$s = \left[\left[\operatorname{arsinh}\frac{1}{2} + \frac{1}{4}\times 1\sqrt{5}\right] - \left[0\right]\right]$$

$$s = \operatorname{arsinh}\frac{1}{2} + \frac{1}{4}\sqrt{5}$$

$$s = \ln\left(\frac{1}{2}+\sqrt{\frac{1}{4}+1}\right) + \frac{1}{4}\sqrt{5}$$

$$s = \ln\left(\frac{1+\sqrt{5}}{2}\right) + \frac{1}{4}\sqrt{5}$$

IYGB - SYNF PAPER B - QUESTION 17

a) If $z = e^{i\theta} = \cos\theta + i\sin\theta$

- $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
- $z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

ADDING THE ABOVE

$$z^n + \frac{1}{z^n} = \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta} = 2\cos n\theta$$

AS REQUIRED

SUBSTRACTING THE ABOVE

$$z^n - \frac{1}{z^n} = \cancel{\cos n\theta} + i\sin n\theta - (\cancel{\cos n\theta} - i\sin n\theta) = 2i\sin n\theta$$

AS REQUIRED

FINALY WE HAVE IF $n=1$

$$(2i\sin\theta)^4 = \left(z - \frac{1}{z}\right)^4$$

$$16\sin^4\theta = z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$$

$$16\sin^4\theta = \left(z^4 + \frac{1}{z^4}\right) - 4\left(z + \frac{1}{z}\right) + 6$$

$$16\sin^4\theta = 2\cos4\theta - 4(2\cos2\theta) + 6$$

$$16\sin^4\theta = 2\cos4\theta - 8\cos2\theta + 6$$

$$8\sin^4\theta = \cos4\theta - 4\cos2\theta + 3$$

AS REQUIRED

IYGB - SYNF PAPER B - QUESTION 17

b) USING PART (a) IN THE TRIGONOMETRIC EQUATION

$$\Rightarrow \quad 8\sin^4\theta + 5\cos 2\theta = 3$$

$$\Rightarrow \quad (\cos 4\theta - 4\cos 2\theta + 3) + 5\cos 2\theta = 3$$

$$\Rightarrow \quad \cos 4\theta + \cos 2\theta = 0$$

$$\Rightarrow \quad (2\cos^2 2\theta - 1) + \cos 2\theta = 0$$

$$\Rightarrow \quad 2\cos^2 2\theta + \cos 2\theta - 1 = 0$$

$$\Rightarrow \quad (2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$$

THUS WE NOW HAVE

$\cos 2\theta = \frac{1}{2}$

$2\theta = \pi/3 \pm 2n\pi$

$2\theta = 5\pi/3 \pm 2n\pi$ $\quad n = 0, 1, 2, 3, \ldots$

$\theta = \pi/6 \pm n\pi$

$\theta = 5\pi/6 \pm n\pi$

$\cos 2\theta = -1$

$2\theta = \pi \pm 2n\pi$ $\quad n = 0, 1, 2, 3, \ldots$

~~$2\theta = \pi \pm 2n\pi$~~

$\theta = \frac{\pi}{2} \pm n\pi$

$\theta =$ ________

$$\therefore \theta = \pi/6, -\frac{5\pi}{6}, 5\pi/6, -\pi/6, \pi/2, -\pi/2$$

$$\theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}$$

IYGB - FURTHER SYNOPTIC PAPER B - QUESTION 18

START BY SKETCHING $y = x^3 - 9x^2 + 24x$.

$$y = x(x^2 - 9x + 24)$$

$$\uparrow$$

$$b^2 - 4ac = 81 - 4 \times 1 \times 24 < 0$$

∴ ONLY INTERCEPT WITH THE CO-ORDINATE AXIS IS $(0,0)$

NEXT LOOK FOR STATIONARY POINTS

$$\frac{dy}{dx} = 3x^2 - 18x + 24$$

$$0 = 3x^2 - 18x + 24$$

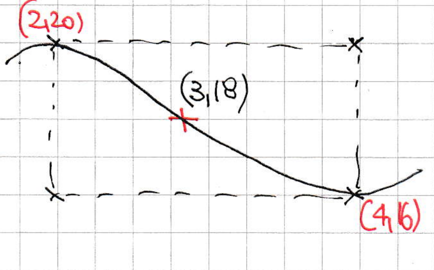
$$0 = x^2 - 6x + 8$$

$$0 = (x-2)(x-4)$$

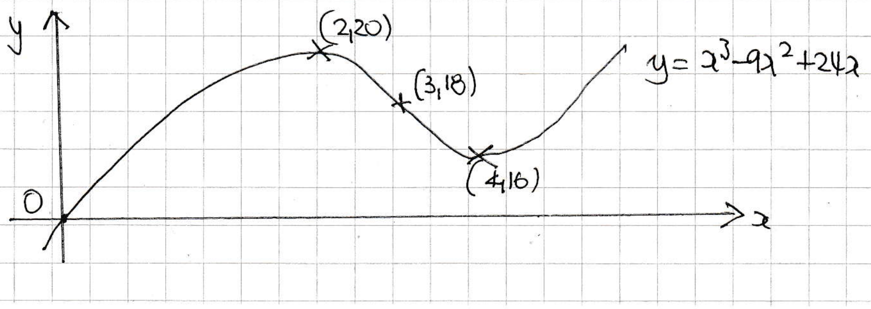
$$x = \begin{cases} 2 \\ 4 \end{cases} \qquad y = \begin{cases} 8 - 9 \times 4 + 48 = 8 + 48 - 36 = 20 \\ 64 - 9 \times 16 + 24 \times 4 = 64 + 96 - 144 = 16 \end{cases}$$

∴ $(2,20)$ & $(4,16)$ ARE STATIONARY — LOCAL MAX & MIN

USING THE ROTATIONAL SYMMETRY PROPERTY OF THE CUBIC, THE NON STATIONARY POINT OF INFLEXION WILL BE AT $(3,18)$

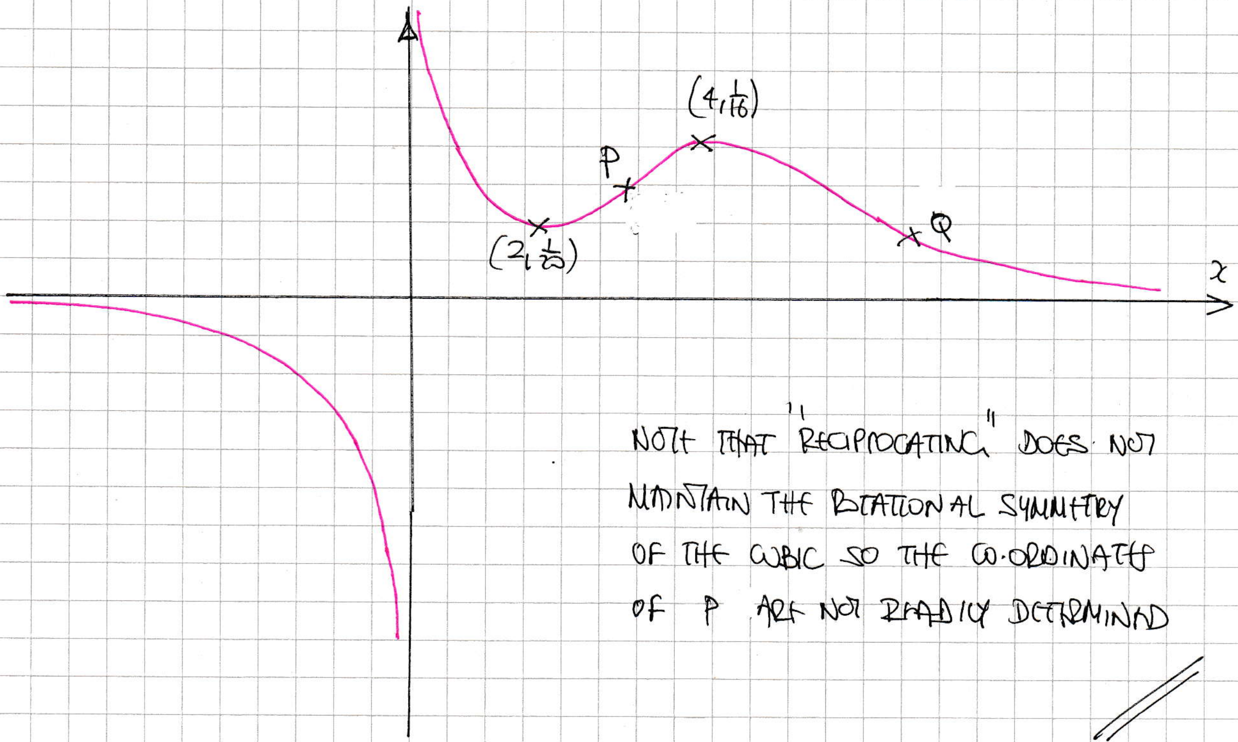


SKETCHING THE CUBIC BEFORE RECIPROCATING



IYGB - FURTHER SYNOPTIC PAPER B - QUESTION 18

"INVERTING THE CUBIC"



NOTE THAT "RECIPROCATING" DOES NOT MAINTAIN THE ROTATIONAL SYMMETRY OF THE CUBIC SO THE CO-ORDINATES OF P ARE NOT READILY DETERMINED //

EVIDENTLY BECAUSE OF THE ASYMPTOTIC BEHAVIOUR AS $x \to +\infty$ THERE IS ANOTHER NON STATIONARY POINT OF INFLEXION AT Q

IYGB - FURTHER SYNOPTIC PAPER B - QUESTION 14

a) BY DIRECT PROOF

$$A^2+B^2+C^2+5 = (8a+1)+(8b+1)+(8c+1)+5$$
$$= 8a+8b+8c+8$$
$$= 8(a+b+c+1)$$

// INDEED A MULTIPLE OF 8

b)
$$A^2(A^2+6)-7 = A^4+6A^2-7$$
$$= (A^2-1)(A^2+7)$$
$$= [(8n+1)-1][(8n+1)+7]$$
$$= 8n(8n+8)$$
$$= 8 \times 8 \times n(n+1)$$
$$= 64 \times 2m$$
$$= 128m$$

// INDEED DIVISIBLE BY 128

c)
$$A^4-B^4 = (A^2-B^2)(A^2+B^2)$$
$$= [(8m+1)-(8n+1)][(8m+1)+(8n+1)]$$
$$= [8m-8n][8m+8n+2]$$
$$= 8(m-n) \times 2(m+n+1)$$
$$= 16(m-n)(m+n+1)$$

// INDEED A MULTIPLE OF 16

IYGB - SYNF PAPER B - QUESTION 20

a) THE LIMIT IS OF THE TYPE "ZERO" × "MINUS INFINITY", SO

$$\lim_{x\to0^+}\left[x^p\ln x\right] = \lim_{x\to0^+}\left[\frac{\ln x}{\frac{1}{x^p}}\right] \quad \leftarrow \text{TYPE } \frac{\text{"}-\infty\text{"}}{\text{"}\infty\text{"}}$$

APPLY L'HOSPITAL RULE AND MANIPULATE FURTHER

$$= \lim_{x\to0^+}\left[\frac{\ln x}{x^{-p}}\right] = \lim_{x\to0^+}\left[\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-p})}\right] = \lim_{x\to0^+}\left[\frac{\frac{1}{x}}{-p\,x^{-p-1}}\right]$$

$$= -\frac{1}{p}\lim_{x\to0^+}\left[\frac{\frac{1}{x}}{\frac{1}{x^{p+1}}}\right] = -\frac{1}{p}\lim_{x\to0^+}\left[\frac{x^{p+1}}{x}\right] = \frac{1}{p}\lim_{x\to0^+}\left[x^p\right] = 0$$

b) PROCEED BY INTEGRATION BY PARTS

$$\int_0^1 x^n\ln x\,dx = \cancel{\left[\frac{1}{n+1}x^{n+1}\ln x\right]_0^1} - \frac{1}{n+1}\int_0^1\frac{x^{n+1}}{x}\,dx$$

$\ln x$	$\frac{1}{x}$
$\frac{1}{n+1}x^{n+1}$	x^n

AT $x=1$ THIS GOES TO ZERO ($\ln 1=0$)

AS $x\to0^+$, $x^{n+1}\ln x \to 0$ (PART a)

$$\therefore \int_0^1 x^n\ln x\,dx = -\frac{1}{n+1}\int_0^1 x^n\,dx$$

$$= -\frac{1}{n+1}\left[\frac{1}{n+1}x^{n+1}\right]_0^1$$

$$= -\frac{1}{n+1}\left[\frac{1}{n+1}(1-0)\right]$$

$$= -\frac{1}{(n+1)^2}$$

IYGB – SYNF PAPER B – QUESTION 21

AS $\frac{d}{dx}(6x^2-x-1) = 12x-1$, USE A SUBSTITUTION

- $u = 6x^2-x-1$
- $\frac{du}{dx} = 12x-1$
- $dx = \frac{du}{12x-1}$
- $6x^2-x+5 = u+6$
- $x=0 \longmapsto u=-1$
- $x=\frac{1}{2} \longmapsto u=0$

$(\frac{3}{2}-\frac{1}{2}-1)$

$$\int_0^{\frac{1}{2}} \frac{12x-1}{(6x^2-x-1)(6x^2-x+5)+10}\,dx$$

$$= \int_{-1}^{0} \frac{\cancel{12x-1}}{u(u+6)+10} \times \frac{du}{\cancel{12x-1}}$$

$$= \int_{-1}^{0} \frac{1}{u^2+6u+10}\,du$$

$$= \int_{-1}^{0} \frac{1}{(u^2+6u+9)+1}\,du$$

$$= \int_{-1}^{0} \frac{1}{(u+3)^2+1}\,du$$

INTEGRATE TO ARCTAN (STANDARD INTEGRAL)

$$= \Big[\arctan(u+3)\Big]_{-1}^{0} = \arctan 3 - \arctan 2$$

SIMPLIFY FURTHER USING THE $\tan(A-B)$ IDENTITY

$$\tan\Big[\arctan 3 - \arctan 2\Big] = \frac{\tan(\arctan 3) - \tan(\arctan 2)}{1 + \tan(\arctan 3)\tan(\arctan 2)}$$

$$= \frac{3-2}{1+3\times 2} = \frac{1}{7}$$

THUS GIVING

$$\dots = \arctan 3 - \arctan 2 = \arctan\frac{1}{7}$$

$(n=7)$

IYGB - SYNF PAPER B - QUESTION 22

ANTICLOCKWISE, ROTATION, ABOUT O, BY $\frac{\pi}{2}$

$$\underline{i} \longmapsto \underline{j} \qquad \begin{pmatrix}1\\0\end{pmatrix} \longmapsto \begin{pmatrix}0\\1\end{pmatrix}$$

$$\underline{j} \longmapsto -\underline{i} \qquad \begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}-1\\0\end{pmatrix} \qquad \therefore \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$$

NEED TO ADD A TRANSLATION BY THE VECTOR $\begin{pmatrix}1\\2\end{pmatrix}$

$$\begin{pmatrix} \boxed{\text{IDENTITY}} & \boxed{\text{TRANSLATION VECTOR}} \\ 0 \quad 0 & 1 \end{pmatrix} = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 1\end{pmatrix}$$

WE CAN "MULTIPLY" OR STATE THE RESULTING MATRIX

IDENTITY — TRANSLATION — ROTATION 90

$$\begin{pmatrix}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & -1 & 1\\1 & 0 & 2\\0 & 0 & 1\end{pmatrix}$$

FINALLY THE INVARIANT POINT

$$\begin{pmatrix}0 & -1 & 1\\1 & 0 & 2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix} = \begin{pmatrix}x\\y\\1\end{pmatrix}$$

OBTAIN 2 NON TRIVIAL EQUATIONS

$$\left.\begin{aligned}-y+1 &= x\\ x+2 &= y\end{aligned}\right\} \Rightarrow \begin{aligned}-y+1+2 &= y\\ 3 &= 2y\\ y &= 3/2 \quad \& \quad x = -\tfrac{1}{2}\end{aligned}$$

$$\therefore \left(-\frac{1}{2}, \frac{3}{2}\right)$$