Oxford Cambridge and RSA

Practice Paper – Set 1

A Level Further Mathematics A

Y545/01 Additional Pure Mathematics

MARK SCHEME

Duration: 1 hour 30 minutes

MAXIMUM MARK 75

DRAFT

This document consists of 11 pages

Text Instructions

1. Annotations and abbreviations

Annotation in scoris	**Meaning**
✓and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Highlighting	
Other abbreviations in mark scheme	**Meaning**
E1	Mark for explaining a result or establishing a given result
dep*	Mark dependent on a previous mark, indicated by *
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working
AG	Answer given
awrt	Anything which rounds to
BC	By Calculator
DR	This question included the instruction: In this question you must show detailed reasoning.

2. Subject-specific Marking Instructions for A Level Further Mathematics A

a Annotations should be used whenever appropriate during your marking. The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks. It is vital that you annotate standardisation scripts fully to show how the marks have been awarded. For subsequent marking you must make it clear how you have arrived at the mark you have awarded.

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct solutions leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly. Correct but unfamiliar or unexpected methods are often signalled by a correct result following an apparently incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, escalate the question to your Team Leader who will decide on a course of action with the Principal Examiner.
If you are in any doubt whatsoever you should contact your Team Leader.

c The following types of marks are available.

M
A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

A
Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B
Mark for a correct result or statement independent of Method marks.

E
Mark for explaining a result or establishing a given result. This usually requires more working or explanation than the establishment of an unknown result.

Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

d When a part of a question has two or more ‘method’ steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation ‘dep*’ is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.

e The abbreviation FT implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only – differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, what is acceptable will be detailed in the mark scheme. If this is not the case please, escalate the question to your Team Leader who will decide on a course of action with the Principal Examiner.
Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be ‘follow through’. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

f Unless units are specifically requested, there is no penalty for wrong or missing units as long as the answer is numerically correct and expressed either in SI or in the units of the question. (e.g. lengths will be assumed to be in metres unless in a particular question all the lengths are in km, when this would be assumed to be the unspecified unit.) We are usually quite flexible about the accuracy to which the final answer is expressed; over-specification is usually only penalised where the scheme explicitly says so. When a value is given in the paper only accept an answer correct to at least as many significant figures as the given value. This rule should be applied to each case. When a value is not given in the paper accept any answer that agrees with the correct value to 2 s.f. Follow through should be used so that only one mark is lost for each distinct accuracy error, except for errors due to premature approximation which should be penalised only once in the examination. There is no penalty for using a wrong value for *g*. E marks will be lost except when results agree to the accuracy required in the question.

g Rules for replaced work: if a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests; if there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others. NB Follow these maths-specific instructions rather than those in the assessor handbook.

h For a genuine misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate’s data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question. Marks designated as cao may be awarded as long as there are no other errors. E marks are lost unless, by chance, the given results are established by equivalent working. ‘Fresh starts’ will not affect an earlier decision about a misread. Note that a miscopy of the candidate’s own working is not a misread but an accuracy error.

i If a calculator is used, some answers may be obtained with little or no working visible. Allow full marks for correct answers (provided, of course, that there is nothing in the wording of the question specifying that analytical methods are required). Where an answer is wrong but there is some evidence of method, allow appropriate method marks. Wrong answers with no supporting method score zero. If in doubt, consult your Team Leader.

j If in any case the scheme operates with considerable unfairness consult your Team Leader.

Question			Answer	Marks	AOs	Guidance
1			$x \equiv 4 \equiv 11, 18, 25, \ldots \pmod 7$ $\Rightarrow x \equiv 25 \pmod 7$ So $x \equiv 25 \pmod 7$ and $x \equiv 25 \pmod{41}$ $\Rightarrow x \equiv 25 \pmod{287}$ since hcf(7, 41) = 1	**M1** **A1** **M1** **A1** **E1** **[5]**	**1.1a** **1.1** **2.1** **2.2a** **2.4**	Considering equivalents to 4 (mod 7) Putting the two together **OR** $x = 287n + 25,\ n \in \mathbb{Z}$
			OR $x = 7a + 4 = 41b + 25$ $\Rightarrow 7(a - 3) = 41b \Rightarrow b = 7c$ since hcf(7, 41) = 1 $\Rightarrow a = 41c + 3 \Rightarrow x = 7(41c + 3) + 4$ i.e. $x = 287c + 25\ (c \in \mathbb{Z})$	**M1** **M1** **E1** **M1** **A1**		 Eliminating either *a* or *b* Obtaining *x* in terms of a single parameter
			OR Reciprocal of $41 \equiv 6 \pmod 7$ is 6 and Reciprocal of 7 (mod 41) is 6 The *Chinese Remainder Theorem* guarantees a unique solution mod 7×41 since hcf(7, 41) = 1 Solution is $x = 287n + r,\ n \in \mathbb{Z}$, where $r = 4\times6\times\frac{287}{7} + 25\times6\times\frac{287}{41} \pmod{287}$ i.e. $r = 2034 \pmod{287} \equiv 25$	**M1 A1** **E1** **M1** **A1**		 Full implementation of the *CRT*
2			E.g. $\mathbf{x} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -9 \end{pmatrix}$, $\mathbf{y} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{z} = \mathbf{d} - \mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ Volume = $\frac{1}{6}\lvert \mathbf{x} \cdot \mathbf{y} \times \mathbf{z} \rvert$ $\begin{vmatrix} 2 & 2 & -9 \\ 5 & 0 & -2 \\ -1 & 1 & 2 \end{vmatrix} = -57$ $\Rightarrow$ Volume = 9.5	**M1 *** **A1** **dep*M1** **B1** **A1** **[5]**	**1.1a** **1.1** **1.1a** **1.1** **2.2a**	Subtraction to find any 3 sides of the tetrahedron Attempted use of formula A correct (non-zero) scalar triple product (possibly **BC**). Condone sign error **FT** det; must be positive final answer

Question			Answer	Marks	AOs	Guidance
3	(i)		$\frac{\partial z}{\partial x} = \frac{1}{y}\sin y - \frac{y}{x}\sin x - \frac{y}{x^2}\cos x$ **oe**	M1 A1	1.1a 1.1	Partial differentiation w.r.t x (including use of the *Product* or *Quotient Rule*)
			$\frac{\partial z}{\partial y} = \frac{x}{y}\cos x - \frac{x}{y^2}\sin y + \frac{1}{x}\cos x$ **oe**	M1 A1 [4]	1.1a 1.1	Partial differentiation w.r.t y (including use of the *Product* or *Quotient Rule*)
	(ii)		When $x = y = \frac{1}{4}\pi$, $z = \sqrt{2}$	B1	1.1	
			$\frac{\partial z}{\partial x} = \frac{-1}{\sqrt{2}}$ and $\frac{\partial z}{\partial y} = \frac{1}{\sqrt{2}}$ **oe**	B1 B1	1.1 1.1	
			Eqn. of tangent-plane is $z = \frac{-1}{\sqrt{2}}(x - \frac{1}{4}\pi) + \frac{1}{\sqrt{2}}(y - \frac{1}{4}\pi) + \sqrt{2}$	M1	2.2a	
			$\Rightarrow x - y + z\sqrt{2} = 2$ **oe**	A1 [5]	1.1	
	(iii)		$\begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix}$ or any suitable multiple	B1 [1]	2.2a	**FT** from their answer to **(ii)**

Question			Answer	Marks	AOs	Guidance
4	(i)	(a)	1, 4, 9, 5, 3	B1 B1 [2]	1.1a 1.1	Give 1st B1 for any 3 correct 2nd B1 for all 5 and allow 0, no incorrect extras condone (0,)1, 4, 9, 5, 3, 3, 5, 9, 4, 1
		(b)	The 5th-powers (mod 11) are 0, 1, 10 so LHS $\equiv$ 0, 1, 10 (mod 11) $2017 \equiv 4 \pmod{11}$ so RHS $\equiv$ 4, 5, 8, 2, 9, 7 (mod 11) Since LHS and RHS are not equal modulo 11, the equation has no solution in integers	M1 A1 M1 A1 E1 [5]	3.1a 1.1 1.1a 2.2a 2.4	Looking at 5th powers mod 11 All 3 found with no extras $2017 \equiv 4 \pmod{11}$ **BC** **FT** each of **(a)**'s residues + 4 Must follow from fully correct working
	(ii)		**DR** mod 9, $M \equiv 2^{12} - 1 = (2^3)^4 - 1 \equiv (-1)^4 - 1 = 1 - 1 = 0$ **OR** $11^{12} - 1 = (11^6 - 1)(11^6 + 1) = (11^3 - 1)(11^3 + 1)(11^6 + 1)$ and $11^3 + 1 = 1332$, which is divisible by 9 using the standard divisibility test	B1	3.1a	Using the *difference-of-two-squares* factorisation (e.g.)
			mod 5, $M \equiv 1^{12} - 1 = 0$ or via above factorisation, noting that $5 \mid 11^3 - 1 = 1330$	B1	3.1a	NB: can argue via $10 \mid (11^n - 1)\ \forall$ integers $n \in \mathbb{N}$ by the *Factor Theorem*; etc.
			The factor $11^6 - 1 \equiv 0 \pmod{7}$ by *FLT* since hcf(11, 7) = 1 and 7 is prime	B1	2.1	
			$M = 11^{12} - 1 \equiv 0 \pmod{13}$ by *FLT* since hcf(11, 13) = 1 and 13 is prime	B1	2.4	
			mod 61, $11^2 = 121 \equiv -1 \Rightarrow 11^{12} - 1 \equiv (-1)^6 - 1 = 1 - 1 = 0$	B1 [5]	2.2a	Any reasonable, clearly explained method is acceptable
			NB $M = 11^{12} - 1 = 3\ 138\ 428\ 376\ 720 = 2^4.3^2.5.7.13.19.37.61.1117$ (Also, M has $11^3 + 1$ as a factor, and $11^3 + 1 = (11 + 1)(11^2 - 11.1 + 1^2)$ by the *sum-of-two-cubes* factorisation, so it follows that M is divisible by $11^2 - 11.1 + 1^2 = 111 = 3\times37$; that is, M is a multiple of 37)			**NB** *FLT* is *Fermat's Little Theorem*

Question			Answer	Marks	AOs	Guidance
5	(i)	(a)	Characteristic Equation is $\lambda^2 - 1.3\lambda - 0.3 = 0$	M1	1.1a	
			$\Rightarrow \lambda = 1.5, -0.2$	A1	1.1	**BC**
			$\Rightarrow$ General Solution is $X_n = A \times 1.5^n + B \times (-0.2)^n$	B1	1.2	**FT** their λs
			Use of $X_0 = 12$ and $X_1 = 1$ to obtain equations in A, B: i.e. $12 = A + B$ & $1 = 1.5A - 0.2B$	M1	1.1a	
			Solving $\Rightarrow A = 2,\ B = 10$	M1	1.1	Eqns. solved simultaneously (e.g. **BC**)
			Solution is $X_n = 2 \times 1.5^n + 10 \times (-0.2)^n$	A1	2.5	**cao** brackets required
				[6]		
		(b)	For large n, $(-0.2)^n \to 0$	B1	3.1b	
			so $X_n \to 2 \times 1.5^n$ which is of the form $a\,r^n$, hence a GP	E1	2.2a	**OR** $3 \times 1.5^{n-1}$, of the form $a\,r^{n-1}$, hence a GP [MUST have some explanation that this is a GP]
				[2]		
	(ii)	(a)	Tabulating the given sequence (or using calculator equation solver)	M1	3.1a	**BC** or manual calculation
			$X_{32} \approx 862\,880 < 1\,000\,000$ and $X_{33} \approx 1\,294\,320 > 1\,000\,000$ so $n = 33$	A1	3.2a	Properly justified
				[2]		
		(b)	$X_n = 2 \times 1.5^n > 1\,000\,000$	M1	2.1	Attempt at solving (logs not essential)
			$\Rightarrow n > \dfrac{\log 500000}{\log 1.5} = 32.36...$ so $n = 33$	A1	1.1	Properly justified
				[2]		
	(iii)	(a)	$X_{n+2} = \text{INT}\left[1.3X_{n+1} + 0.3X_n\right]$	B1	3.3	
				[1]		
		(b)	$X_{n+2} = \text{INT}\left[1.3X_{n+1} + 0.3X_n + 1\right]$ **or** $X_{n+2} = \text{INT}\left[1.3X_{n+1} + 0.3X_n\right] + 1$	B1	3.5c	
				[1]		
		(c)	If $1.3X_{n+1} + 0.3X_n$ were *exactly* an integer value at any stage, then the next X_n would be too large by 1	B1	3.5b	
				[1]		

Question			Answer	Marks	AOs	Guidance
6	(i)		$I_1 = \int_0^{\sqrt{3}} t\sqrt{1+t^2}\,\mathrm{d}t = \left[\frac{1}{3}\left(1+t^2\right)^{\frac{3}{2}}\right]_0^{\sqrt{3}}$	M1	1.1a	By "recognition" (reverse *Chain Rule*) integration or any other suitable method (e.g. by substitution): MUST have $k(1 + t^2)^{1.5}$
			$= \frac{1}{3}(8-1) = \frac{7}{3}$	A1	1.1	**AG** from correct working
				[2]		
	(ii)		$I_n = \int_0^{\sqrt{3}} t^n\sqrt{1+t^2}\,\mathrm{d}t = \int_0^{\sqrt{3}} t^{n-1}.t\sqrt{1+t^2}\,\mathrm{d}t$	M1	2.1	Correct splitting and attempt at *integration by parts*
			$= \left[t^{n-1}\,\frac{1}{3}\left(1+t^2\right)^{\frac{3}{2}}\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}}(n-1)t^{n-2}.\frac{1}{3}\left(1+t^2\right)^{\frac{3}{2}}\,\mathrm{d}t$	A1 A1	1.1, 1.1	One for each correct part
			$\Rightarrow 3I_n = \left(\sqrt{3}\right)^{n-1}.8 - 0 - (n-1)\int_0^{\sqrt{3}} t^{n-2}\left(1+t^2\right)\sqrt{1+t^2}\,\mathrm{d}t$	M1	3.1a	Splitting up the power of $(1 + t^2)$ suitably
			$\Rightarrow 3I_n = 8\left(\sqrt{3}\right)^{n-1} - (n-1)\{I_{n-2} + I_n\}$	A1	1.1	2nd integral correctly identified in terms of *I*'s
			$\Rightarrow (n+2)\,I_n = 8\left(\sqrt{3}\right)^{n-1} - (n-1)\,I_{n-2}$	A1	2.2a	**AG** Legitimately obtained from correct rearrangement of *I* terms
				[6]		
	(iii)		$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = (30t^2)^2 + (30t)^2$	M1	1.1a	Attempted use of formula
			$= 900t^2(1 + t^2)$ **soi**	A1	1.1	Correct, any form
			$A = 2\pi\int_0^{\sqrt{3}} 15t^2.30t\sqrt{1+t^2}\,\mathrm{d}t$	M1	2.1	Formula used correctly with relevant terms in t
			$= 900\pi I_3$ i.e. $k = 900$, $m = 3$	A1	1.1	
			Use of $n = 3$ in Reduction Formula: $5\,I_3 = 8\left(\sqrt{3}\right)^2 - 2\times\frac{7}{3} \Rightarrow I_3 = \frac{58}{15}$	M1	3.1a	
			so that $A = 3480\pi$	A1	2.2a	**cao**
				[6]		

Question			Answer	Marks	AOs	Guidance
7	(i)		$\mathbf{X} = \frac{1}{\sqrt{3}}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow \mathbf{X}^2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},\ \mathbf{X}^3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix},$ $\mathbf{X}^4 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},\ \mathbf{X}^5 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix},\ \mathbf{X}^6 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ Note that $\mathbf{X}^7 = -\mathbf{X}$ etc. so that none of $\mathbf{X}^7$ to $\mathbf{X}^{11}$ is equal to **I** $\Rightarrow \mathbf{X}^{12} = \mathbf{I}$ and so $n(M) = 12$ because the powers of **X** now cycle every 12th power but no sooner	M1 A1 A1 E1 [4]	1.1a 1.1 2.2a 2.4	Attempt at powers of **X** (possibly **BC**) Correct up to $\mathbf{X}^6$ (NB – not *all* needed) Candidates may calculate each of these **BC** Correctly deduced Clear statement needed to justify $n(M) = 12$
	(ii)		The set is closed (each element is one of the 12 powers of **X**); matrix multiplication is known to be associative; the multiplicative identity **I** is in M; and the inverse of $\mathbf{X}^k$ is $\mathbf{X}^{12-k}$ for all suitable k … so we have a group G The group is cyclic since it is generated by **X** (or $\mathbf{X}^5$ or $\mathbf{X}^7$ or $\mathbf{X}^{11}$) **oe**	B1 B1 B1 B1 E1 [5]	2.1 1.1 1.1a 1.1 2.4	One for each group axiom Inverses may be explicitly paired-up Only one generator needs to be mentioned here
	(iii)		Proper subgroups of G are $H_2 = \{\mathbf{X}^6, \mathbf{I}\}$, $H_3 = \{\mathbf{X}^4, \mathbf{X}^8, \mathbf{I}\}$, $H_4 = \{\mathbf{X}^3, \mathbf{X}^6, \mathbf{X}^9, \mathbf{I}\}$, $H_6 = \{\mathbf{X}^2, \mathbf{X}^4, \mathbf{X}^6, \mathbf{X}^8, \mathbf{X}^{10}, \mathbf{I}\}$ The above subgroups are those generated as follows: $H_2 = (\mathbf{X}^6)$, $H_3 = (\mathbf{X}^4) = (\mathbf{X}^8)$, $H_4 = (\mathbf{X}^3) = (\mathbf{X}^9)$ and $H_6 = (\mathbf{X}^2) = (\mathbf{X}^{10})$ and are thus unique Mention that *Lagrange's Theorem* states that $\|H\| \mid \|G\|$ so there are no subgroups of orders 5, 7, 8, 9, 10 or 11	B1 B1 B1 E1 B1 [5]	3.1a 1.1 3.1a 3.1a 2.2a	Any two correct; All four correct (+ no extras – ignore inclusion of {**I**} and/or G) Condone omission to mention that any subgroup containing $\mathbf{X}^5$ or $\mathbf{X}^7$ or $\mathbf{X}^{11}$ is G (since they are generators)
			OR For listing the subgroups (as above) The elements have orders {12, 6, 4, 3, 12, 2, 12, 3, 4, 6, 12, 1} and one can argue from the orders of elements that (e.g.) : a subgroup of order 2 must contain **I** and an element of order 2 $\Rightarrow H_2$ uniquely; a subgroup of order 3 must contain **I** and an element of order 3 $\Rightarrow H_3$ uniquely, since there are only two such elements and they are an inverse-pair; similarly for H_6 Then as above for no subgroups of orders 5, 7, 8, 9, 10 or 11	B1 B1 B1 E1 B1		

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