UK Maths Trust

Junior Mathematical Challenge

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Solutions and investigations

1 May 2025

These solutions augment the shorter solutions also available online. The solutions given here are full solutions, as explained below. In some cases we give alternative solutions. There are also many additional problems for further investigation. We welcome comments on these solutions. Please send them to challenges@ukmt.org.uk.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that occasionally you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can sometimes be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. Therefore here we have aimed at giving full solutions with all steps explained (or, sometimes, left as an exercise). We hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Junior Mathematical Olympiad and similar competitions).

Enquiries about the Junior Mathematical Challenge should be sent to:

challenges@ukmt.org.uk *www.ukmt.org.uk*

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	A	E	A	C	D	D	B	B	D	E	B	B	A	B	C	C	B	E	B	B	D	D	A	D

1. What is $\frac{1}{2}$ of 10% of 20 ?

A 1 B 2 C 5 D 10 E 20

SOLUTION **A**

10% is $\frac{1}{10}$th. Therefore 10% of 20 is $\frac{1}{10} \times 20 = 2$.

Therefore $\frac{1}{2}$ of 10% of 20 is $\frac{1}{2}$ of 2. Hence $\frac{1}{2}$ of 10% of 20 is 1.

FOR INVESTIGATION

1.1 What is $\frac{1}{3}$ of 20% of 60?

1.2 $\frac{1}{n}$ of 25% of 80 is 4. What is the value of n?

1.3 $\frac{1}{n^2}$ of n% of 1000 is 2. What is the value of n?

2. What is the value of $\frac{4444 + 5555}{1111 + 2222}$?

A 3 B 6.5 C 33 D 333 E 3333

SOLUTION **A**

It is not difficult to see that the expression given in the question simplifies to $\frac{4+5}{1+2}$ and that this equals 3. It takes rather more space to write out the calculation in full.

$$\begin{aligned}\frac{4444 + 5555}{1111 + 2222} &= \frac{4 \times 1111 + 5 \times 1111}{1 \times 1111 + 2 \times 1111}\\ &= \frac{(4+5) \times 1111}{(1+2) \times 1111}\\ &= \frac{4+5}{1+2}\\ &= \frac{9}{3}\\ &= 3.\end{aligned}$$

FOR INVESTIGATION

2.1 What are the values of (a) $\frac{4444 \times 5555}{1111 \times 2222}$ and (b) $\frac{4444 \div 5555}{1111 \div 2222}$?

3. How many of the following are factors of 2025?

15	25	45	75	225

A 1 B 2 C 3 D 4 E 5

SOLUTION **E**

It can be checked that

$$2025 = 3 \times 3 \times 3 \times 3 \times 5 \times 5 \text{ and } 225 = 3 \times 3 \times 5 \times 5.$$

It follows that

$$2025 = (3 \times 3 \times 5 \times 5) \times (3 \times 3) = 225 \times 9.$$

Hence 225 is a factor of 2025.

In a similar way it follows that $15 = 3 \times 5$, $25 = 5 \times 5$, $45 = 3 \times 3 \times 5$ and $75 = 3 \times 5 \times 5$ are also factors of 2025.

Therefore all five of the given options are factors of 2025.

FOR INVESTIGATION

3.1 Check that $2025 = 3 \times 3 \times 3 \times 3 \times 5 \times 5$, and that $225 = 3 \times 3 \times 5 \times 5$.

3.2 Show that 15, 25, 45 and 75 are all factors of 225. [Note that it follows that they are all factorsof 2025.]

3.3 How many factors does 2025 have altogether, including 1 and the number 2025 itself?

> If you know the prime factorization of a positive integer, there is quick way to deduce how many factors the integer has.
>
> Let n be a positive integer with $n > 1$. Then n has a unique prime factorization $p^a \times q^b \times r^c \times \ldots$, where $p, q, r \ldots$ are primes with $p < q < r < \ldots$, and $a, b, c \ldots$ are positive integers.
>
> Then the factors of n are all the integers
>
> $$p^i \times q^j \times r^k \times \ldots \qquad (*)$$
>
> where $0 \leq i \leq a$, $0 \leq j \leq b$, $0 \leq k \leq c$ and so on. Hence i can take $a + 1$ different values, j can take $b + 1$ values, c can take $k + 1$ values and so on.
>
> Hence there are
>
> $$(a + 1) \times (b + 1) \times (c + 1) \times \ldots \qquad (**)$$
>
> numbers of the form (*). Therefore the formula (**) gives the number of factors of n.
>
> For example, we have seen that $2025 = 3^4 \times 5^2$. So here $a = 4$ and $b = 2$. Hence the number of factors of 2025 is given by $(4 + 1) \times (2 + 1) = 5 \times 3 = 15$.

3.4 How many factors does the number 1 000 000 have altogether?

3.5 Give an example of a positive integer with 6 factors.

3.6 Find the smallest positive integer which has 6 factors.

3.7 Which positive integers have an odd number of factors?

4. In March 2024, the British runner, Jasmin Paris, became the first woman ever to finish the 100-mile ‘Barkley Marathons’ in Tennessee. Her finishing time of 59 hours, 58 minutes and 21 seconds meant that she narrowly beat the cut-off time of 60 hours for the event. By how many seconds did Jasmin beat this cut-off time?

A 99 B 109 C 119 D 129 E 139

Solution **A**

21 seconds is 60 − 21 seconds, that is, 39 seconds, short of 1 minute. Hence 58 minutes and 21 seconds is 39 seconds short of 59 minutes. 59 minutes is 1 minute short of 1 hour. Hence 59 hours, 58 minutes and 21 seconds is 1 minute and 39 seconds short of 60 hours.

1 minute and 39 seconds is 60 + 39 seconds = 99 seconds.

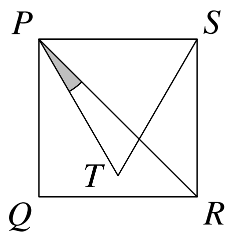
Therefore 59 hours, 58 minutes and 21 seconds is 99 seconds short of 60 hours.

For investigation

4.1 What was Jasmin's average speed in miles per hour? Give your answer
(a) correct to 1 decimal place, and, using a calculator,
(b) correct to 6 decimal places.

5. The diagram shows the square $PQRS$ and the equilateral triangle PTS.
What is the size of the angle RPT?

A $10°$ B $12°$ C $15°$ D $22\frac{1}{2}°$ E $30°$



Solution **C**

Because $PQRS$ is a square, in the triangle PSR we have $\angle PSR = 90°$ and $PS = SR$. It follows that $\angle RPS = \angle PRS$.

The angles in a triangle add up to 180°. It follows that $\angle RPS = \frac{1}{2}(180 - 90)° = \frac{1}{2}(90)° = 45°$.

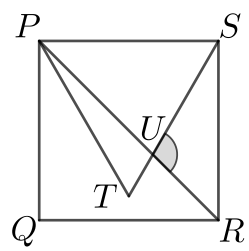
Because PTS is an equilateral triangle, $\angle SPT = 60°$.

Therefore $\angle RPT = \angle SPT - \angle RPS = 60° - 45° = 15°$.

For investigation

5.1 Let U be the point where PR meets ST.

What is the size of the angle RUS?



6. Carol and Sandra swim at their local 25 m pool. Sandra always swims 4 lengths in the same amount of time as Carol swims 3 lengths. One Monday morning, Carol swims 36 lengths. How far would Sandra swim in the same time?

A 600 m B 675 m C 900 m D 1200 m E 1350 m

SOLUTION **D**

$\frac{4}{3} \times 36 = 48$. Therefore, in the time during which Carol swims 36 lengths, Sandra swims 48 lengths.

Therefore the distance that Sandra swims is $48 \times 25\,\text{m} = 1200\,\text{m}$.

FOR INVESTIGATION

6.1 On one Tuesday Sandra swam 800 m. How many lengths did Carol swim that morning?

7. What is the difference between the smallest four-digit multiple of 3 and the largest three-digit multiple of 4?

A 3 B 4 C 5 D 6 E 7

SOLUTION **D**

We use the convenient facts that

(a) The test for an integer to be a multiple of 3 is that the sum of its digits is a multiple of 3.

(b) The test for an integer to be multiple of 4 is that the its last two digits (its tens and units digits) form a number that is a multiple of 4.

You are asked to show why these tests work in Problem 7.2.

Using test (a), we see that neither 1000 nor 1001 is a multiple of 3, but 1002 is a multiple of 3. Therefore 1002 is the smallest four-digit multiple of 3.

Using test (b), we see that neither 999, nor 998, nor 997 is a multiple of 4, because 99, 98 and 97 are not multiples of 4. However $96 = 4 \times 24$ and so 996 is a multiple of 4. Therefore 996 is the largest three-digit multiple of 4.

$1002 - 996 = 6$. Therefore 6 is the difference between the smallest four-digit multiple of 3 and the largest three-digit multiple of 4.

FOR INVESTIGATION

7.1 What is the difference between the smallest four-digit multiple of 7 and the largest three-digit multiple of 11?

7.2 Show that the tests given above for an integer to be a multiple of 3, and for an integer to be a multiple of 4 are correct.

7.3 (a) Find a test for whether an integer is divisible by 8.

(b) Use this test to determine whether the number 246 842 684 is divisible by 8.

8. The country which, on average, eats the most chocolate per person is Switzerland, with the average person consuming 8.8 kg each year. How many 100 g chocolate bars would this be?

A 8.8 B 88 C 880 D 8800 E 88 000

SOLUTION **B**

8.8 kg = 8800 g and 8800 ÷ 100 = 88. Therefore the number of 100 g chocolate bars eaten by the average person in Switzerland each year is 88.

FOR INVESTIGATION

8.1 It is estimated that the total annual consumption of chocolate in the UK is 500 000 tonnes. Approximately how many kilograms of chocolate per head is this?

9. All four digits of 2 two-digit numbers are different. What is the smallest possible sum of the two numbers?

A 30 B 33 C 37 D 40 E 42

SOLUTION **B**

We first want the sum of the tens digits of the two numbers to be as small as possible. The tens digit of a two-digit number cannot be 0. Therefore the smallest possible sum of these digits is $1 + 2 = 3$.

We also want the sum of the units digits to be as small as possible. Since 1 and 2 have already been used as digits, the smallest possible sum of the units digits is $0 + 3 = 3$.

The sum of 2 two-digit numbers whose units and tens digits both have sum 3 is $30 + 3 = 33$. Therefore this is the smallest possible sum.

FOR INVESTIGATION

9.1 Give an example of 2 two-digit numbers, all of whose digits are different, with sum 33.

9.2 What is the largest possible sum of 2 two-digit numbers, all of whose digits are different?

10. The length of a turtle is 60 cm plus a third of its length. How long is the turtle?

A 70 cm B 75 cm C 85 cm D 90 cm E 120 cm

SOLUTION **D**

Since the length of a turtle is 60 cm plus a third of its length, 60 cm is two thirds of the length of a turtle. Hence 30 cm is one third of the length of a turtle. Therefore the total length of a turtle is 3×30 cm = 90 cm.

FOR INVESTIGATION

10.1 The length of another turtle is 60 cm plus a quarter of its length. How long is this turtle?

11. Three consecutive one-digit positive integers are a square, a cube and a prime (not necessarily in that order). What is the product of the three integers?

A 0 B 6 C 60 D 210 E 504

SOLUTION **E**

The only one-digit positive cubes are 1 and 8. Therefore either 1 or 8 occurs in the sequence that we need to find.

Therefore, the sequence is either 1, 2, 3 or 6, 7, 8 or 7, 8, 9.

In the sequence 1, 2, 3, we have 1, which is a square and a cube, followed by two primes. So this sequence does not consist of a square, a cube and a prime, in some order.

The sequence 6, 7, 8 does not include a square.

However the sequence 7, 8, 9 consists of a prime, a cube and a square, in this order. Hence this is the required sequence.

The product of the three integers in this sequence is $7 \times 8 \times 9 = 7 \times 72 = 504$.

FOR INVESTIGATION

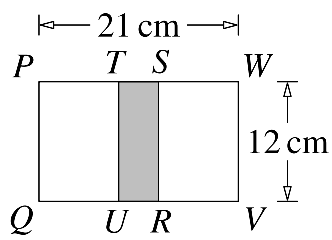
11.1 Find the smallest positive integer which is neither a square, nor a cube nor a prime.

11.2 Find the smallest positive integer which is neither a square, nor a cube, nor a triangular number nor a prime.

12. Two identical squares, $PQRS$ and $TUVW$, overlap to form the 21 cm by 12 cm rectangle $PQVW$ shown.

What is the area, in cm^2, of the shaded rectangle $TURS$?

A 24 B 36 C 48 D 72 E 96



SOLUTION **B**

The rectangle has height 12 cm. Therefore the side length of the squares is also 12 cm.

It follows that $SW = PW - PS = 21\,\text{cm} - 12\,\text{cm} = 9\,\text{cm}$.

Hence $TS = TW - SW = 12\,\text{cm} - 9\,\text{cm} = 3\,\text{cm}$.

Therefore the rectangle $TURS$ has height 12 cm and width 3 cm. Therefore the area of this rectangle is $(12 \times 3)\,\text{cm}^2$, that is, $36\,\text{cm}^2$.

FOR INVESTIGATION

12.1 Suppose that two identical squares $PQRS$ and $TUVW$ overlap as in the diagram of this question, but with side lengths different to those in Question 12. Suppose also that PW has length 24 cm and the area of the shaded region $TURS$ is one third of the area of $PQVW$.

In this case what is the length of WV?

13. In a sequence of rectangles, the first has height 10 cm and width 13 cm. Each of the following rectangles is 2 cm higher and 1 cm wider than the previous rectangle in the sequence.
How many rectangles in this sequence are also squares?

A 0 B 1 C 2 D 4 E an infinite number

SOLUTION **B**

METHOD 1

The first rectangle in the sequence is 3 cm wider than it is high. In each subsequent rectangle this difference reduces by 1 cm. Therefore the fourth rectangle in the sequence has the same height as its width and is therefore a square. All the subsequent rectangles are higher than they are wide.

Therefore there is just one square in the sequence.

METHOD 2

The first rectangle has width 13 cm. Each subsequent rectangle is 1 cm wider than its predecessor. So the sequence of widths is 13 cm, 14 cm, 15 cm, ... Thus the width of the nth rectangle in the sequence is $(12 + n)$ cm.

The first rectangle has height 10 cm. Each subsequent rectangle is 2 cm higher than its predecessor. So the sequence of heights is 10 cm, 12 cm, 14 cm, ... Thus the height of the nth rectangle in the sequence is $(8 + 2n)$ cm.

The nth rectangle in the sequence is a square provided that its height is equal to its width, that is, provided that $8 + 2n = 12 + n$. This equation has just the one solution $n = 4$. Therefore there is just one square in the sequence, namely, the fourth rectangle in the sequence.

FOR INVESTIGATION

13.1 Show that there is no rectangle in the sequence which is twice as high as it is wide.

14. Which of these is equal to $3^3 + 4^3 + 5^3$?

A 6^3 B 8^3 C 10^3 D 12^3 E 12^9

SOLUTION **A**

We have

$$3^3 + 4^3 + 5^3 = 27 + 64 + 125 = 91 + 125 = 216$$

and

$$6^3 = 6 \times 6 \times 6 = 36 \times 6 = 216.$$

Therefore the correct option is A.

15. Gill has had a baby girl! Weighing the baby at the clinic was not as much of a problem as when Gill was small, but the only available scales measured in imperial units. Baby weighed 8 pounds 13 ounces. Given that there are 16 ounces in a pound and 1 ounce is equivalent to approximately 28.35 grams, what was the approximate weight, in kilograms, of Gill's baby ?

A 3 kg B 4 kg C 5 kg D 6 kg E 7 kg

Solution **B**

8 pounds is $8 \times 16 = 128$ ounces. Therefore 8 pounds and 13 ounces is $128 + 13 = 141$ ounces. This is approximately equal to 141×28.35 grams.

We now need to find the approximate value of 141×28.35. We do this by replacing 141 and 28.35 by rounded values. The given options are not very far apart, and so we should not make the approximation too rough. If we round 141 down to 140 and round 28.35 up to 30, we have rounded 141 down by less than 1% and we have rounded 28.35 up by rather more than 5%. So we should expect our answer to be an overestimate, around 4% more than the actual value..

We have

$$141 \times 28.35 \approx 140 \times 30 = 4200.$$

Hence Gill's baby weighed approximately 4200 g. This is 4.2 kg. Hence option B is a good approximation to the baby's weight.

A Short Biography of Gill

Gill is older than the UKMT as she dates back to the first *Schools Mathematical Challenge* organised by Tony Gardiner in 1988. Question 14 in that year was all about weighing her when she was a baby, but her name was not revealed to be Gill until the third *Challenge* in 1990 when Gill was two years old*.

Gill has appeared frequently in Challenge questions since the UKMT was established in 1996. From these we can piece together a sketchy outline of her life so far.

She sat the Junior Mathematical Challenge in 2002, moved to a new house in 2003, and celebrated her 18th birthday with a meal in a restaurant in 2006.

She passed her driving test in 2014 and moved into a new flat in 2015.

Junior Mathematical Challenge questions in 2016, 2017 and 2018 tell us that Gill was then a Mathematics teacher, and now we know that she has had a baby girl.

Is this her only child, and what is the baby's name? Perhaps the answers to these questions will be revealed in future Challenge questions.

* *Mathematical Challenge,* Tony Gardiner, Cambridge University Press, 1996.

For investigation

15.1 A *gill* is a unit of volume. There are 4 gills in a pint, and 8 pints in a gallon. In the UK a gallon is approximately $4\frac{1}{2}$ litres. Approximately, how many cubic centimetres are there in a gill?

16. What is the sum of 20% of 40% of 60 and 40% of 60% of 80?

A 20 B 20.4 C 24 D 24.4 E 28

SOLUTION **C**

We can do the calculation needed to answer this question by either working with fractions or decimals, for example, putting $40\% = \frac{40}{100}$, or $40\% = 0.40$.

With either method it pays to gather terms and take out common factors before working out the values of the expressions we obtain.

METHOD 1

$$\begin{aligned} 20\% \text{ of } 40\% \text{ of } 60 + 40\% \text{ of } 60\% \text{ of } 80 &= \left(\frac{20}{100} \times \frac{40}{100} \times 60\right) + \left(\frac{40}{100} \times \frac{60}{100} \times 80\right) \\ &= \frac{20 \times 40 \times 60 + 40 \times 60 \times 80}{100 \times 100} \\ &= \frac{40 \times 60 \times (20 + 80)}{100 \times 100} \\ &= \frac{40 \times 60 \times 100}{100 \times 100} \\ &= \frac{40 \times 60}{100} \\ &= \frac{2400}{100} = 24. \end{aligned}$$

METHOD 2

$$\begin{aligned} 20\% \text{ of } 40\% \text{ of } 60 + 40\% \text{ of } 60\% \text{ of } 80 &= 0.20 \times 0.40 \times 60 + 0.40 \times 0.60 \times 80 \\ &= 0.40 \times (0.20 \times 60 + 0.60 \times 80) \\ &= 0.40 \times (12 + 48) \\ &= 0.4 \times 60 = 24. \end{aligned}$$

FOR INVESTIGATION

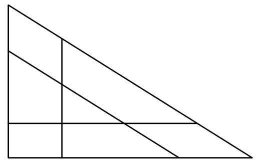
16.1 Express both 20% of 40% of 60 and 40% of 60% of 80 as simplified fractions. Hence check that their sum is 24,

16.2 x and y are numbers with the property that 20% of 40% of x = 40% of 60% of y.

Express y in terms of x.

17. The diagram shows three pairs of parallel line segments. Altogether, how many triangles are there in the diagram?

A 4 B 6 C 8 D 10 E 12

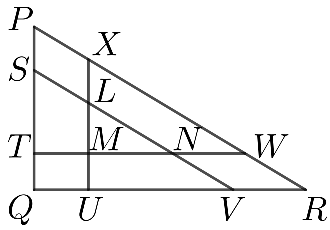


SOLUTION **C**

METHOD 1

We list the triangles in a systematic way, so that each triangle is counted, and is counted once only.

We have labelled the points in the diagram as shown.

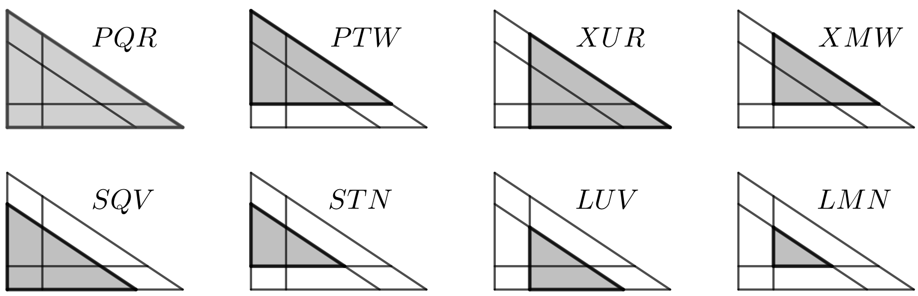


We list the triangles in the diagram according to their topmost vertex.

It can be seen that the only points that are the top vertex of a triangle are P, X, S and L.

Furthermore, each of these points is the top vertex of two triangles. It follows that there are eight triangles in the diagram. They are PQR, PTW, XUR, XMW, SQV, STN, LUV and LMN.

These triangles are shown in the diagram below.



METHOD 2

A triangle is formed from three lines no two of which are parallel. Three such lines may be chosen by picking one from each of the three pairs of parallel lines in the diagram. This may be done in $2 \times 2 \times 2 = 8$ ways. Therefore there are 8 triangles in the diagram.

FOR INVESTIGATION

17.1 How many parallelograms are there in the diagram of this question?

17.2 How many trapeziums are there in the diagram of this question?

17.3 How many quadrilaterals are there in the diagram of this question?

17.4 Given three pairs of parallel lines, what is the maximum number of (a) triangles, (b) parallelograms, (c) trapeziums, and (d) quadrilaterals, they can form?

18. A particular number has exactly nine factors, including 1 and itself.
The number has 9 and 12 as factors. What is the number?

A 24 B 36 C 48 D 72 E 144

SOLUTION **B**

We use the fact, explained in Problem 3.1, that if the positive integer n has the prime factorization given by

$$n = p^{\alpha} \times q^{\beta} \times r^{\gamma} \times \ldots,$$

where p, q, r, ... are distinct prime numbers, and α, β, γ, ... are positive integers, then n has

$$(\alpha + 1)(\beta + 1)(\gamma + 1) \ldots \quad (*)$$

factors, including 1 and n.

Let n be the number that we seek.

Since 9 and 12 are factors of n the primes 2 and 3 are factors of n. Hence n has at least two prime factors.

Since n has exactly 9 factors, we have

$$(\alpha + 1)(\beta + 1)(\gamma + 1) \cdots = 9, \qquad (1)$$

where the prime factorization of n is given by $n = p^{\alpha} \times q^{\beta} \times r^{\gamma} \times \ldots$, as in the box above.

Since $\alpha, \beta, \gamma \ldots$ are all positive integers, the factors in the product (1) are all at least 2.

The only way 9 can be expressed as the product of more than one integer, each greater than 1, is as 3×3. Hence, the product (1) consists of just the two factors $(\alpha + 1)(\beta + 1)$, with $\alpha = \beta = 2$.

It follows that $n = 2^2 \times 3^2 = 4 \times 9 = 36$.

FOR INVESTIGATION

18.1 Find the smallest positive integer which has 90 as one of its factors and 24 factors in total.

18.2 Which is the largest 3-digit positive integer that has exactly 9 factors?

18.3 How many factors does the integer 10! have?

Note: 10!, pronounced *ten factorial,* is the product of all the integers from 1 to 10, inclusive. That is, $10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$.

More generally, for each positive integer n, $n!$ is the product of all the integers from 1 to n, inclusive.

The value of $n!$ grows very rapidly as n increases. Therefore a standard calculator which only displays ten digits will only give all the digits of $n!$ for low values of n.

18.4 Find the largest value of n for which your calculator displays all the digits when you use it to find the value of $n!$.

18.5 Find the largest value of n for which your calculator does not give an error message when you use it to find the value of $n!$ because this value is outside the calculator's range. Normally this will be the largest integer n for which $n! < 10^{100}$.

19. Ibraheem has 10 cards. Four display the letter P; three display Q; two display R and one displays S. He also has 10 counters. One shows the number 3; two show 1; three show 4 and four show 2. He places one counter on the top of each card so that every pairing is different. Which one of these pairings does he *not* make?

A P2 B R4 C Q1 D R2 E S4

SOLUTION **E**

Our first quick method assumes that Ibraheem can complete the task, and that there is only one of the parings given as options that he does not make.

The second method shows that these assumptions are correct.

METHOD 1

Ibraheem pairs the four counters showing the number 2 with cards displaying different letters. Therefore he pairs the four counters showing the number 2 with one each of the cards displaying the letters P, Q, R and S.

Therefore Ibraheem pairs the only card displaying S with the counter showing the number 2. Hence the pairing that Ibraheem does not make is S4.

METHOD 2

Ibraheem has 10 cards which display the following letters

P P P P Q Q Q R R S

and 10 counters which show the following numbers

1 1 2 2 2 2 3 4 4 4.

Ibraheem has four cards displaying the letter P which he needs to pair with four different numbers. So he pairs one P with each of the numbers 1, 2, 3 and 4. This leaves him with the following unpaired letters Q Q Q R R S, and with the following numbers not yet used 1 2 2 2 4 4.

He next pairs the three Qs with each of the numbers 1, 2, and 4. Then he pairs the two Rs with 2 and 4, leaving the S to be paired with 2.

Therefore Ibraheem ends up with the following pairings

P1, P2, P3, P4, Q1, Q2, Q4, R2, R4, S2.

It may now be seen that, of the given options, the pairing that Ibraheem does not make is S4.

FOR INVESTIGATION

19.1 List all the other possible pairings that Ibraheem does *not* make.

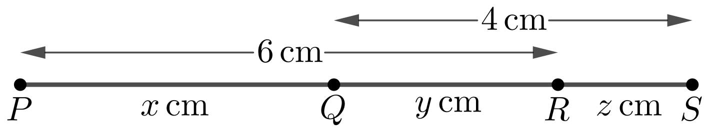
20. P, Q, R and S are four points in that order on a straight line; $PR = 6$ cm, $QS = 4$ cm and R is 1 cm nearer to S than it is to Q. What is the length of PS?

A 7 cm B 7.5 cm C 8 cm D 8.5 cm E 9 cm

SOLUTION **B**

We let the lengths of PQ, QR and RS be x cm, y cm and z cm, respectively.

Then the length of PS which we are required to find is $(x + y + z)$ cm.



Since $PR = 6$ cm,

$$x + y = 6. \qquad (1)$$

Since $QS = 4$ cm,

$$y + z = 4. \qquad (2)$$

Since R is 1 cm nearer to S than it is to Q,

$$y - z = 1. \qquad (3)$$

By subtracting equation (3) from equation (2), we obtain

$$2z = 3$$

from which it follows that

$$z = 1.5. \qquad (4)$$

It follows from equations (1) and (4) that

$$\begin{aligned} x + y + z &= 6 + 1.5 \\ &= 7.5. \end{aligned}$$

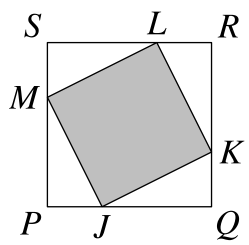
Therefore the length of PS is 7.5 cm.

FOR INVESTIGATION

20.1 Find the lengths of PQ and QR.

21. In the diagram, $PQRS$ and $JKLM$ are squares. Each corner of $JKLM$ is one third of the way along an edge of $PQRS$. What is the ratio of the area of $JKLM$ to the area of $PQRS$?

A 1 : 2 B 5 : 9 C 3 : 5 D 4 : 7 E 3 : 4



Solution **B**

We let the side length of the square $PQRS$ be $3s$. [Note that we use $3s$ to avoid fractions.]

It follows that the area of the square $PQRS$ is $(3s)^2$, that is, $9s^2$.

Since each corner of $JKLM$ is one third of the way along the edges of $PQRS$, we have $PJ = s$ and $PM = 2s$.

Because $PQRS$ is a square $\angle MPJ = 90°$. So we can apply Pythagoras' Theorem to the triangle MPJ. In this way we obtain

$$\begin{aligned} MJ^2 &= PJ^2 + PM^2 \\ &= s^2 + (2s)^2 \\ &= s^2 + 4s^2 \\ &= 5s^2. \end{aligned}$$

It follows that the area of the square $JKLM$ is $5s^2$.

Therefore the ratio of the area of $JKLM$ to the area of $PQRS$ is $5s^2 : 9s^2$. This is equal to 5 : 9.

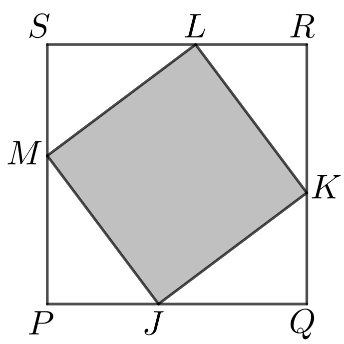
For investigation

21.1 In the diagram $PQRS$ and $JKLM$ are squares.

(a) Prove that the ratios $PJ : JQ$, $QK : KR$, $RL : LS$ and $SM : MP$ are all equal.

(b) Suppose that the ratio of the area of $JKLM$ to the area of $PQRS$ is 25 : 49, and that the common value of the ratios $PJ : JQ$, $QK : KR$, $RL : LS$ and $SM : MP$ is $\lambda : 1$, with $\lambda < 1$.

What in this case is the value of λ ?



22. Idil has 12 bags of sweets. Some bags contain 3 mints, 4 toffees and a fudge; some bags contain 4 mints, 5 toffees and 2 fudges; the remaining bags contain 6 mints and 3 fudges. The bags contain 31 toffees in total. In total, how many fudges do the bags contain?

A 22 B 23 C 24 D 25 E 26

Solution **D**

Suppose that Idil has

x bags containing 3 mints, 4 toffees and a fudge,
y bags containing 4 mints, 5 toffees and 2 fudges, and
z bags containing 6 mints and 3 fudges.

Note that x, y and z are all positive integers.

Also, note that the bags contain a total of $x + 2y + 3z$ fudges.

Since Idil has 12 bags of sweets,

$$x + y + z = 12. \qquad (1)$$

Since Idil has 31 toffees in total,

$$4x + 5y = 31. \qquad (2)$$

It follows from (2) that $x = \dfrac{31 - 5y}{4}$.

Since x is a positive integer, $31 - 5y$ is a positive multiple of 4.

It can be checked that the only integer value of y for which this true is 3. When $y = 3$, we have $x = \dfrac{31 - 5 \times 3}{4} = \dfrac{16}{4} = 4$.

It now follows from (1) that $z = 12 - x - y = 12 - 3 - 4 = 5$.

Since the number of fudges is $x + 2y + 3z$, we conclude that the number of fudges in the bags is $1 \times 4 + 2 \times 3 + 3 \times 5 = 4 + 6 + 15 = 25$.

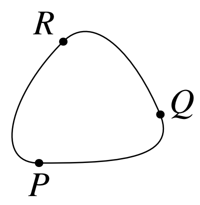
For investigation

22.1 Check that the only integer value of y for which $\dfrac{31 - 5y}{4}$ is a positive integer is given by $y = 3$.

22.2 Idil recounted the number of toffees that she had, and found that she had miscounted the first time. She actually had 32 toffees.

How many fudges did she have?

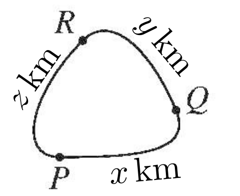
23. The map in the diagram shows three towns and the roads between them. The length of the road between any two towns is a whole number of kilometres.
The route from P to R via Q is twice as long as the direct route from P to R. The route from P to Q via R is three times as long as the direct route from P to Q. Which of the following could be the length of the route from P back to P via Q and R?



A 74 km B 81 km C 95 km D 108 km E 124 km

SOLUTION **D**

Let the direct routes from P to Q, from Q to R, and from R to P have lengths x km, y km and z km, respectively, as shown in the diagram on the right.



We are told that x, y and z are positive integers.

The length of the route from P back to P via Q and R is $(x + y + z)$ km.

From the information given in the question, we have

$$x + y = 2z \quad (1) \qquad \text{and} \qquad z + y = 3x \quad (2).$$

By equation (1), $x + y + z = 3z$ and is therefore a multiple of 3.

By equation (2) $x + y + z = 4x$ and is therefore a multiple of 4.

It follows that $x + y + z$ is a multiple of both 3 and 4, and hence it is a multiple of 12. Hence, of the given options, the only possible length of the route is 108 km.

FOR INVESTIGATION

23.1 Suppose that the length of the route from P back to P via Q and R is 108 km. Find the values of x, y and z in this case.

23.2 Suppose that, as in the question, the length of the road between any two towns is a whole number of kilometres, the route from P to R via Q is twice as long as the direct route from P to R, and the route from P to Q via R is three times as long as the direct route from P to Q.

What is the shortest possible distance from P back to P via Q and R?

23.3 Suppose that, as in the question, the length of the road between any two towns is a whole number of kilometres, but the route from P to R via Q is three times as long as the direct route from P to R, and the route from P to Q via R is four times as long as the direct route from P to Q.

What is the shortest possible distance from P back to P via Q and R?

24. When 180 is divided by a positive integer N, the remainder is 5.
For how many values of N is this true?

A 4 B 5 C 6 D 7 E 8

SOLUTION **A**

Suppose that the integer N has the property that when 180 is divided by N the remainder is 5.

It follows that 180 is equal to a multiple of N plus 5. We can express this by the equation

$$180 = kN + 5, \text{ where } k \text{ is an integer.} \qquad (1)$$

(1) is equivalent to

$$175 = kN, \text{ where } k \text{ is an integer.} \qquad (2)$$

(2) is equivalent to saying that N is a factor of 175.

When you divide an integer by N, the remainder will be at most $N - 1$. Therefore, to obtain remainder of 5, we must have $N \geq 6$.

Therefore the positive integers, N, such that when 180 is divided by N, the remainder is 5, are the positive integers that are factors of 175 and greater than 5.

The prime factorization of 175 is $5^2 \times 7^1$. Therefore 175 has $3 \times 2 = 6$ factors [see the solution to Question 3]. Of these, 1 and 5 are not greater than 5, but the other four factors are greater than 5.

Therefore there are four positive integers that leave remainder 5 when divided into 180.

FOR INVESTIGATION

24.1 List the four positive integers that leave remainder 5 when divided into 180.

24.2 How many positive integers leave remainder 5 when divided into 365?

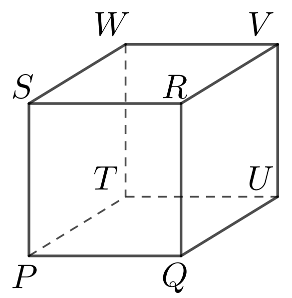
25. Given a cube, how many equilateral triangles are there whose vertices are three vertices of the cube?

A 1 B 3 C 6 D 8 E 24

Solution **D**

An equilateral triangle has three edges of equal length. All its angles have size 60°.

We have labelled the vertices of the cube, as shown in the diagram on the right.

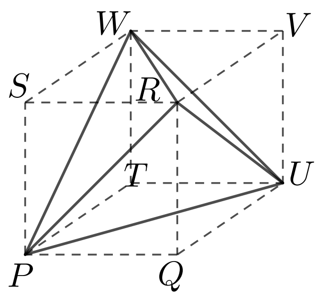


The lines joining the vertices of the cube fall into three different classes according to their lengths.

The edges of the cube all have the same length. However two of these edges that meet, always meet at an angle of 90°. Therefore they cannot form two of the edges of an equilateral triangle.

The lines joining the diagonally opposite vertices of any face of the cube are all the same length. We will call these *face diagonals.* The face diagonals are longer than the edges.

Each vertex is an end point of three face diagonals. For example, P is an end point of the face diagonals PR, PU and PW. Each pair of these face diagonals forms two edges of an equilateral triangle whose third edge is also a face diagonal. Thus there are three of these equilateral triangles PRU, PRW and PUW that have P as one of its vertices.

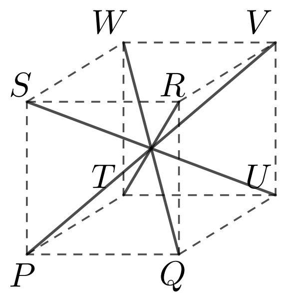


The same holds for each of the 8 vertices of the cube.

This apparently gives rise to $8 \times 3 = 24$ equilateral triangles whose edge are face diagonals. However, this calculation counts each of these triangles three times, once for each of its vertices.

Hence there are $24 \div 3 = 8$ equilateral triangles of this type.

There are also the longer lines which form space diagonals of the cube, joining opposite vertices. There are four of these lines PV, QW, RT and SU. No two of these have a vertex in common.



Therefore there is no equilateral triangle with three of these lines as its edges.

We conclude that there are 8 equilateral triangles that have three vertices of the cube as its vertices.

For investigation

25.1 How many right-angled triangles are there whose vertices are three vertices of a given cube?

25.2 How many isosceles triangles are there whose vertices are three vertices of a given cube?