Junior Mathematical Olympiad 2019

The Junior Mathematical Olympiad (JMO) has long aimed to help introduce able students to (and to encourage them in) the art of problem-solving and proof. The problems are the product of the imaginations of a small number of volunteers writing for the JMO problems group. After the JMO, model solutions for each problem are published on the UKMT website and are included in the UKMT Yearbook.

However, these neatly printed solutions usually represent only one approach (often where several are feasible) and convey none of the sense of investigation, rough-working or false-starts that usually precede the finished article.

In marking JMO scripts, markers want to encourage complete (and concise) proofs in section B but, in doing so, are content to accept less than perfectly written answers, as long as it is clear where the proof is going – we appreciate that there is a limited time available, and that to polish and to present takes the place of considering another problem. Nevertheless, clarity and insight are what are looked for.

We have collected below a small number of solutions submitted for the JMO in 2019, with the aim that future candidates can see what some students in this age group do achieve and that they might aspire to emulate it. In many ways they are ordinary solutions, not brilliantly and startlingly original – but mathematically they are to be commended, in particular, for the logical progression from one point to the next, and for clear presentation.

It is not easy to generalise about what makes a good solution or how a candidate can achieve success in the JMO, but there are a few points to ponder on:

- using trial and improvement to a large degree is generally considered not very mathematical – it will lend little insight into the structure of the problem, and even if you get an answer that works, there should always be a concern that it is, perhaps, not the only answer;
- if you do have to resort to calculating your way through a large number of cases, then the calculations should be shown as part the solution – the reader should not have to make suppositions about what you have tried (or not tried);
- diagrams should be large enough to contain all subsequent details – however, the use of just one diagram can convey very little of the order of the proof, especially in geometrical questions;
- avoid long sentences, particularly those which frequently use *if* and *could be*.
- the use of algebra makes it possible to express connections in simple ways, where a huge number of words would be otherwise necessary;
- if algebra is used, variables should be fully declared;
- attempt as many questions as you can do really well in – it is better, in terms not only of scoring marks but also of honest satisfaction, to spend your time concentrating on a few questions and providing full, clear and accurate solutions, rather than to have a go at everything you can, and to achieve not very much in any of them. Out of the 1000 students or so who take part in the JMO, generally only a handful achieve full marks in all six section B questions.

If this all sounds rather negative, it has, in balance, to be said that JMO candidates produce an astonishing amount of work that is worthy of high praise. We hope that the work below will give future candidates a flavour of what is to be encouraged, in style, method and presentation.

B1. In this word-sum, each letter stands for one of the digits 0–9, and stands for the same digit each time it appears. Different letters stand for different digits. No number starts with 0.

Find all the possible solutions of the word-sum shown here.

$$\begin{array}{r} J\,M\,O \\ J\,M\,O \\ +\ J\,M\,O \\ \hline I\,M\,O \end{array}$$

Solution:

B1) We know that since no number can start with 0, $J \neq 0$ and $I \neq 0$.

So from the sum, we can see that when a single number is added three times (multiplied by 3), the units digit must ~~to~~ = O.

Using this chart that shows us possible digits, their number when multiplied by 3, when 1 is added to this multiplication (if 1 must be carried over) and when 2 is added to the multiplication (if 2 must be carried over). 3 cannot be carried over, as the largest digit, 9, × 3 = 27, so two is the highest carry over.

Digit	×3	(×3)+1	(×3)+2
0	0	1	2
1	3	4	5
2	6	7	8
3	9	10	11
4	12	13	14
5	15	16	17
6	18	19	20
7	21	22	23
8	24	25	26
9	27	28	29

We can see there are only two options for O, O = 0 or O = 5, as both end with the same digit (0 and 15 respectively)

<u>If O = 5</u>

If O = 5, then we know 1 must be ~~to~~ carried over. So in this case, the second addition of M will be 3M+1 = yM, where y = ~~any~~ a digit from 0 - 2 inclusive.

B1 continued:

However, looking at the chart, no digit that is multiplied by 3 and then had 1 added on has the corrosponding digit as M. So the ~~therony~~ sum cannot be continued, and the theory that O = 5 is invalid

If O = 0

If O = 0, then there is no carry over onto the tens column. This means the second sum of the Ms is 3M = yM, where y = ~~any integ~~ ~~any~~ a digit between 0 - 2 inclusive.
There is only one possible value for M in this case, which is M = 5, as 5 × 3 = 15. (~~the digit 0 does not w~~ though the digit 0 does work too, it is already used a the letter O, so cannot be used for a different term).

This means that 1 is carried over to the hundreds column which means the sum of J's is 3J + 1 = I, proving that I can be different to J, but cannot exceed 9, as it will become a two digit number, which is invalid.

This means that there are only 2 options, where J = 1 or J = 2 (J ≠ 3 because (3 × 3) + 1 = 9 + 1 = 10, which is a two digit number.

So, overall, there are 2 × 1 = 2 possible solutions

150	250
150	250
150	250
450	750

B2. The product $8000 \times K$ is a square, where K is a positive integer.

What is the smallest possible value of K?

Solution:

B2) If 8000K is a square, you must be able to split the prime factors of 8000K into 2 equal groups, each containing the same prime numbers.

8000 can be written as a product of its prime factors as

$$2^6 \times 5^3 = 2 \times 2 \times 2 \times 5 \times 5 \times 2 \times 2 \times 2 \times 5.$$

2^6 can be written as $2^3 \times 2^3$. 5^3 can be written as $5^2 \times 5$ so we must add another 5 so it becomes $5^2 \times 5^2$.

Therefore, the smallest possible value of K is 5.

B3. It takes one minute for a train travelling at constant speed to pass completely through a tunnel that is 120 metres long. The same train, travelling at the same constant speed, takes 20 seconds from the instant its front enters the tunnel to it being completely inside the tunnel.

How long is the train?

Solution

Let the length of the train be l metres.

By the first statement, if the train starts with its front entering the tunnel, 120 metres later its front will exit the tunnel. l metres later than that the back will exit the tunnel.

This takes 60 seconds, so the speed is $\frac{120+l}{60}$ (1)

By the second statement, l metres after the front enters, the back will also enter.

This takes 20 seconds, so the speed is $\frac{l}{20}$ (2)

By (1), (2), as in both circumstances, the train travels at the same speed, so

$$\frac{l}{20} = \frac{120+l}{60}$$

Multiply by 60 throughout:

$$3l = 120 + l$$

Rearrange:

$$2l = 120$$

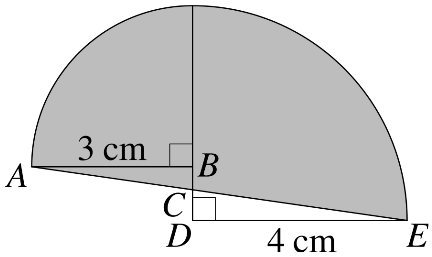
So $$l = 60$$

Hence the train is 60 metres long.

B4. The diagram alongside shows two quarter-circles and two triangles, ABC and CDE. One quarter-circle has radius AB, where $AB = 3$ cm. The other quarter-circle has radius DE, where $DE = 4$ cm.

The area enclosed by the line AE and the arcs of the two quarter-circles is shaded.

What is the total shaded area, in cm^2?



Solution:

Total Area = 3 cm + 4 cm + 3 cm − 4 cm

Area of 3 cm $= \frac{1}{4} \times 3^2 \times \pi = \frac{9\pi}{4} = 2\frac{1}{4}\pi \text{ cm}^2$

Area of 4 cm $= \frac{1}{4} \times 4^2 \times \pi = \frac{16\pi}{4} = 4\pi \text{ cm}^2$

$BD = 1$ cm and $ab : DE$ therefore $BC : CD$

$3 : 4$ $\quad$ $3 : 4$

$BC = \frac{3}{7}$ cm

$CD = \frac{4}{7}$ cm

Area of 3 cm $\frac{3}{7}$ cm $= \frac{1}{2} \times 3 \times \frac{3}{7} = \frac{9}{14} \text{ cm}^2$

Area of $\frac{4}{7}$ 4 $= \frac{1}{2} \times 4 \times \frac{4}{7} = \frac{16}{14} \text{ cm}^2$

Total Area $= 2\frac{1}{4}\pi + 4\pi + \frac{9}{14} - \frac{16}{14}$

$= 6\frac{1}{4}\pi - \frac{1}{2}$

$6\frac{1}{4}\pi - \frac{1}{2} \text{ cm}^2$

B5. My 24-hour digital clock displays hours and minutes only.

How many displayed times in a 24-hour period contain at least one occurrence of the digit 5?

Solution:

B5. We know that there are two hours where the digit 5 always occurs - from 05:00 to 05:59 and from 15:00 to 15:59. Therefore there are at least 120 occurences.

Furthermore, there are 15 occurences of the number 5 in every other hour at the minutes 05, 15, 25, 35, 45 and 50 - 59. There are 22 such hours so 22 × 15 = 330 and 330 + 120 = 450.

Therefore, there are 450 occurences of the number 5.

B6. An equilateral triangle is divided into smaller equilateral triangles.

The diagram on the left shows that it is possible to divide it into 4 equilateral triangles. The diagram on the right shows that it is possible to divide it into 13 equilateral triangles.

What are the integer values of n, where $n > 1$, for which it is possible to divide the triangle into n smaller equilateral triangles?

Solution:

B6

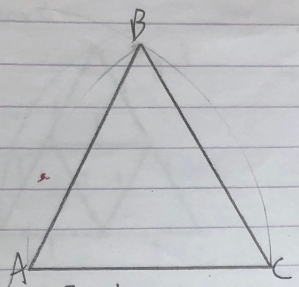


Fig. 1

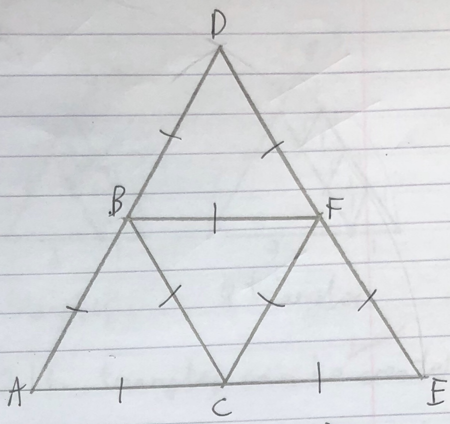


Fig. 2

As the angles of an equilateral triangle are 60°, the angles of the main triangle must also be angles of any smaller triangles. As there are 3 of these, there are at least 3 smaller triangles.

If these 3 triangles took up all of the area, then all of the points along the edge would lie in at least one of the triangles.

So, in Fig. 2, as $\overline{AB} = \overline{AC}$ and $\overline{AD} = \overline{AE}$, so $\overline{BD} = \overline{CE}$. Using the same logic, eventually we get $\overline{AB} = \overline{DF} = \overline{CE}$. This leaves space for a fourth equilateral triangle, ΔBFC.

So the smallest number of triangles obtained is 4.

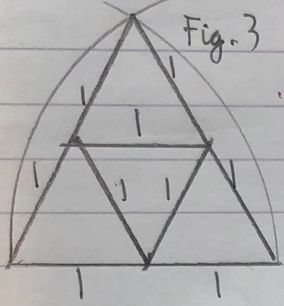
Suppose ΔABC in Fig. 1 is split into k smaller triangles

Hence ΔADE in Fig. 2 is split into $k+3$ smaller triangles as $\Delta ABC \to k$ triangles, $\Delta BDF \to 1$, $\Delta BFC \to 1$, $\Delta CFE \to 1$

B6 continued:

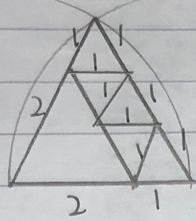
observe

Fig. 3



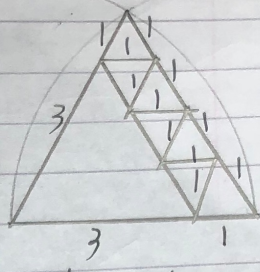
4 triangles

Fig. 4



6 triangles

Fig. 5



8 triangles

In Fig. 3, by (1), 4, 7, 10, 13... triangles can be created.

In Fig. 4, by (1) 6, 9, 13, 16... triangles can be created.

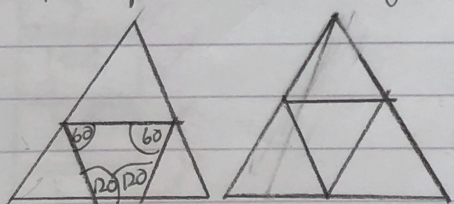
In Fig. 5, by (1) 8, 11, 14, 17... triangles can be created.

Hence 4, 6, 7... triangles can be created.

So $n=4$ and $n \geq 6$ are valid values of n.

Now we consider 5 triangles.

There are always 3 triangles at the vertices of the main triangle. The centre must now split into 2 triangles that are equilateral. Only a quadrilateral and triangle can be split into triangles.



If a quadrilateral remains, only a 60 − 60 − 120 − 120 quad can be left. As a 60° angle can't be cut through to make an equilateral triangle, no cuts can be made (so fail). If a triangle is left, it can't be split into 2 equilateral triangles as above. Hence $n \neq 5$.

$\boxed{\text{Ans: } n=4 \text{ or } n \geq 6}$