Junior Mathematical Olympiad 2018

The Junior Mathematical Olympiad (JMO) has long aimed to help introduce able students to (and to encourage them in) the art of problem-solving and proof. The problems are the product of the imaginations of a small number of volunteers writing for the JMO problems group. After the JMO, model solutions for each problem are published on the UKMT website and are included in the UKMT Yearbook.

However, these neatly printed solutions usually represent only one approach (often where several are feasible) and convey none of the sense of investigation, rough-working or false-starts that usually precede the finished article.

In marking JMO scripts, markers want to encourage complete (and concise) proofs in section B but, in doing so, are content to accept less than perfectly written answers, as long as it is clear where the proof is going – we appreciate that there is a limited time available, and that to polish and to present takes the place of considering another problem. Nevertheless, clarity and insight are what are looked for.

We have collected below a small number of solutions submitted for the JMO in 2018, with the aim that future candidates can see what some students in this age group do achieve and that they might aspire to emulate it. In many ways they are ordinary solutions, not brilliantly and startlingly original – but mathematically they are to be commended, in particular, for the logical progression from one point to the next, and for clear presentation.

It is not easy to generalise about what makes a good solution or how a candidate can achieve success in the JMO, but there are a few points to ponder on:

- using trial and improvement to a large degree is generally considered not very mathematical – it will lend little insight into the structure of the problem, and even if you get an answer that works, there should always be a concern that it is, perhaps, not the only answer;
- if you do have to resort to calculating your way through a large number of cases, then the calculations should be shown as part the solution – the reader should not have to make suppositions about what you have tried (or not tried);
- diagrams should be large enough to contain all subsequent details – however, the use of just one diagram can convey very little of the order of the proof, especially in geometrical questions;
- avoid long sentences, particularly those which frequently use *if* and *could be*.
- the use of algebra makes it possible to express connections in simple ways, where a huge number of words would be otherwise necessary;
- if algebra is used, variables should be fully declared;
- attempt as many questions as you can do really well in – it is better, in terms not only of scoring marks but also of honest satisfaction, to spend your time concentrating on a few questions and providing full, clear and accurate solutions, rather than to have a go at everything you can, and to achieve not very much in any of them. Out of the 1000 students or so who take part in the JMO, generally only a handful achieve full marks in all six section B questions.

If this all sounds rather negative, it has, in balance, to be said that JMO candidates produce an astonishing amount of work that is worthy of high praise. We hope that the work below will give future candidates a flavour of what is to be encouraged, in style, method and presentation.

B1. Polly Garter had her first child on her 20th birthday, her second child exactly two years later, and her third child exactly two years after that.

How old was Polly when her age was equal to the sum of her three children's ages?

An algebraic solution:

B1 Let x be the amount of years since Polly Garter's 20th birthday. Polly will be $20+x$ years old. Her first child would be x years old since it was born that day. Her second child would be $x-2$ years old and her last child would be $x-4$ years old.

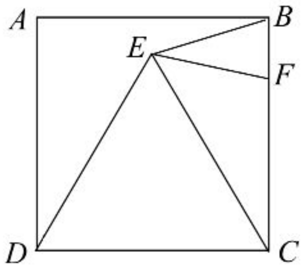
$$20+x=(x)+(x-2)+(x-4)$$
$$20+x=3x-6$$
$$26+x=3x$$
$$2x=26$$
$$x=13$$

Since 13 years have passed since her 20th birthday she is
$20+13=33$ years old

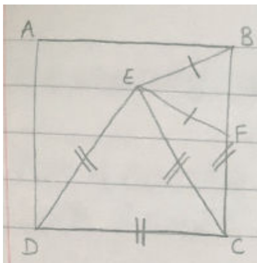
A word-based solution:

1) When Polly was age 24, she had her 3rd child. At this point the combined age of her children was 6. Given there are 3 children, every year the combined age of the children goes up by 3, whereas Polly's age goes up by one. This means they catch up by 2 years every year. If she is 24 and her kids together are 6, they have 18 years to catch up. This will take them 9 years which will mean that Polly is 33 when her kids combined age is the same as hers.

B2. In the diagram shown, $ABCD$ is a square and point F lies on BC. Triangle DEC is equilateral and $EB = EF$.
What is the size of $\angle CEF$?



Solution



In the diagram, $AB = BC = CD = DA$, as they are all sides of a square.
Also, $CD = DE = EC$, as they are all sides of an equilateral triangle.

Since $CD = CE$, and $CD = BC$, CE must be equal to CB, so $\triangle BCE$ is isosceles.

$\angle ECD$ is 60°, because the angles in an equilateral triangle are equal and add up to 180°.

All the angles in a square are right angles, so $\angle BCD = 90^\circ$.

$\angle BCE = \angle BCD - \angle ECD = 90^\circ - 60^\circ = 30^\circ$.

As $\triangle BCE$ is isosceles, ~~ang~~ and angles opposite equal sides of an isosceles triangle are equal, it follows that $\angle CBE = \angle CEB$.

Angles in a triangle add up to 180°, so $\angle CBE = \angle CEB = (180^\circ - 30^\circ) \div 2 = 150^\circ \div 2 = 75^\circ$.

We know that $EB = EF$, so $\triangle EBF$ is also isosceles, with $\angle EBF = \angle EFB$.

But $\angle EBF$ is the same as $\angle EBC$, since F lies on BC.

So $\angle EBF = \angle EFB = 75^\circ$.

Again, using the fact that angles in a triangle add up to 180°, $\angle BEF = 180^\circ - (75^\circ \times 2) = 180^\circ - 150^\circ = 30^\circ$.

$\angle CEF = \angle CEB - \angle FEB = 75^\circ - 30^\circ = 45^\circ$ $\therefore$ $\underline{\underline{\angle CEF = 45^\circ}}$

B3. The letters a, b and c stand for non-zero digits. The integer 'abc' is a multiple of 3; the integer '$cbabc$' is a multiple of 15; and the integer '$abcba$' is a multiple of 8.

What is the integer 'abc'?

Solution

Since 'abc' is a multiple of 3, its digits must equal a multiple of three as well. Also, since 'cbabc' is a multiple of 15, it must be a multiple of 5 and 3, so 'c' must equal 5 or 0, but since $a, b, c \neq 0$, $c = 5$. $a+b+c$ is already a multiple of 3, so for $c+b+a+b+c$ to be a multiple of 3, $c+b$ must also be a multiple of 3. $\therefore$ b could be 1, 4 or 7.

The number 'abcba' is a multiple of 8, so its last 3 digits must be a multiple of 8 as well. $\therefore$ 'cba' is a multiple of 8. We've already established that $a+b+c$ is a multiple of 3, so 'cba' must also be a multiple of 3. Also, since 3 and 8 don't share any common factors other than 1, 'cba' must be a multiple of 24, which is 3×8. The only multiples of 24 in the fine hundreds are:

504, 528, 552, 576

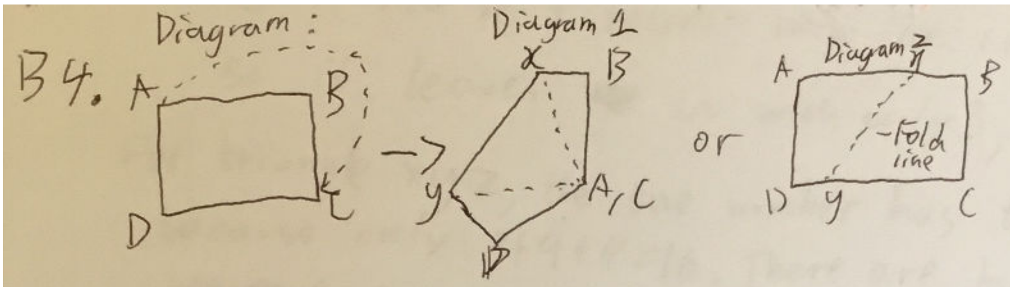
In only one of them, $b = 1, 4$ or 7. $\therefore$ 'cba' = 576 and 'abc' = $\underline{\underline{675}}$

B4. A rectangular sheet of paper is labelled *ABCD*, with *AB* one of the longer sides. The sheet is folded so that vertex *A* is placed exactly on top of the opposite vertex *C*. The fold line is *XY*, where *X* lies on *AB* and *Y* lies on *CD*.

Prove that triangle *CXY* is isosceles.

Solution

B4.



$\because$ xy is a fold,

$\therefore$ ∠Axy in Diagram one = ∠Axy in Diagram 2

$\because$ ABCD is a rectangle.

$\therefore$ AB // DC

$\therefore$ ∠Axy in Diagram 2 = ∠xyc in Diagram 2 = ∠xyc in Diagram 1

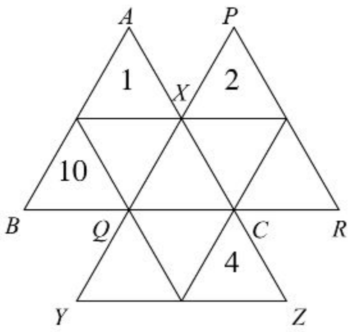
$\therefore$ ∠Axy in Diagram 1 = ∠xyc in Diagram 1

$\therefore$ Cxy triangle is iscoseles

B5. The diagram shows three triangles, ABC, PQR and XYZ, each of which is divided up into four smaller triangles. The diagram is to be completed so that the positive integers from 1 to 10 inclusive are placed, one per small triangle, in the ten small triangles. The totals of the numbers in the three triangles ABC, PQR and XYZ are the same.

Numbers 1, 2, 4 and 10 have already been placed.

In how many different ways can the diagram be completed?



Solution

Let the number in the middle be x. When we find the sum of the numbers in <u>each</u> triangle, x will be counted thrice. x belongs to the numbers 1–10. So, if we find the sum of the numbers in each triangle and add them up, we'll get

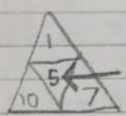
$1+2+3+4+5+6+7+8+9+10+2x = 55+2x$

$\boxed{\text{because } x \text{ belongs to } 1\text{–}10}$

The sum of the numbers in each triangle is equal, so $55+2x$ must be a multiple of 3.

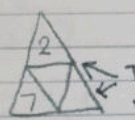
The only value of x that satisfies this is 7. Then, $55+2x = 55+2(7) = 69$. The sum of the numbers in each triangle is $69 \div 3 = 23$.

$\triangle ABC$ looks like this:



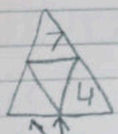
This number must be $23-10-1-7=5$

$\triangle PQR$ looks like this:



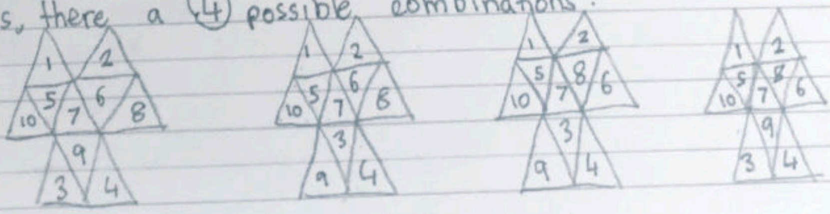
The sum of these must be $23-7-2=14$. The numbers we have left are: 3, 6, 8, 9. The only 2 that make 14 is 6 and 8.

$\triangle XYZ$ looks like this:

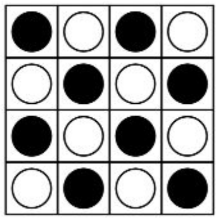


The sum of these is 12, using the only numbers lef 9 and 3.

Thus, there a 4 possible combinations:



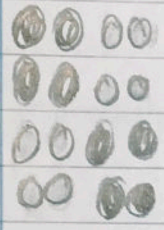
B6. Sixteen counters, which are black on one side and white on the other, are arranged in a 4 by 4 square. Initially all the counters are facing black side up. In one 'move', you must choose a 2 by 2 square within the square and turn all four counters over once. Describe a sequence of 'moves' of minimum length that finishes with the colours of the counters of the 4 by 4 square alternating (as shown in the diagram).



Two explanations of the same set of moves:

B6)

For A4 and D1 to become white the grids C1 : D2 and A3 : B4 must be turned over. Once turned, there is no point in turning these grids again, as A4 must remain white, and cannot be changed via another grid.

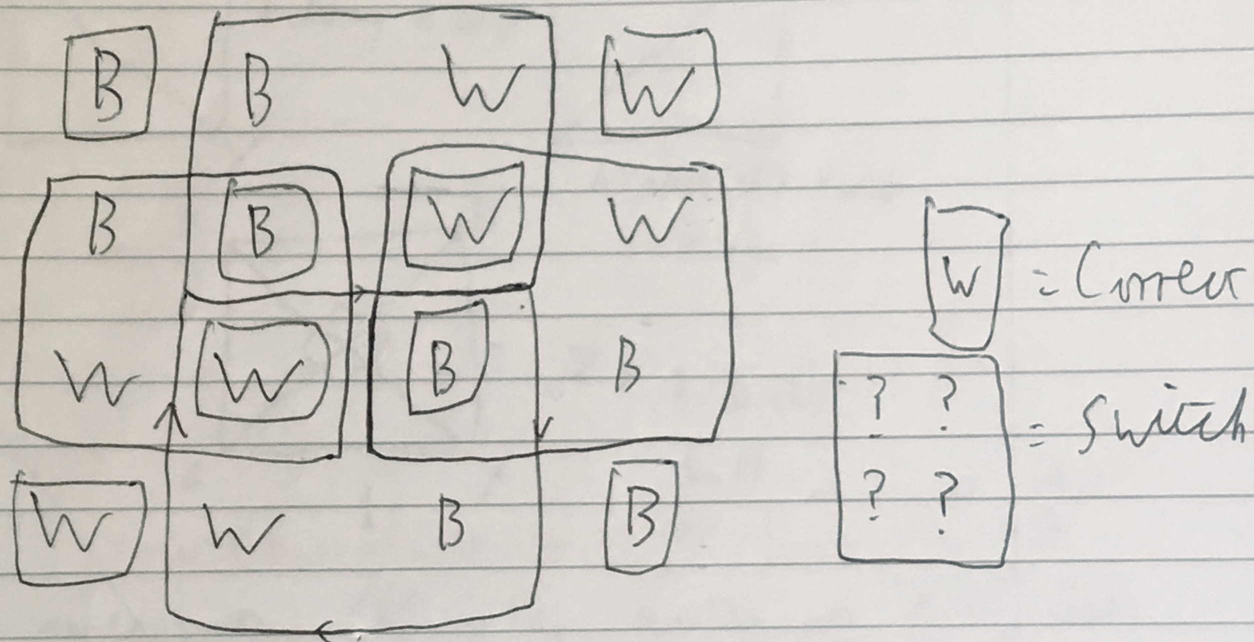


Similarly, grids A1 : B2 and C3 : D4 should not be turned

Now, both B1 and C1 need to be turned. As neither A1 : B2 nor C1 : D2 can be turned, B1 : C2 must be. Similarly, B3 : C4, A2 : B3, and C2 : D3 must be turned. In result:

This has been done in 6 moves. As, at no point did we repeat a move, or use an unnecessary move, it is the minimum length

First the corners must be correct, 2 are wrong, so in 2 moves they are changed leaving:



All the outside (excluding corners) must be switched. Switching all of them keeps the middle ones the same, each one being switched twice, this finishes the pattern.

It can't be less, as it takes at least 2 moves for the corners, and 4 for the rest of the sides, conveniently at the end of this 6 moves the pattern is complete.