UK Maths Trust

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email challenges@ukmt.org.uk
web www.ukmt.org.uk

Junior Mathematical Olympiad 2024

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

The 2024 paper

General comments

This year's paper turned out to be one of the easier papers in recent years, roughly similar to last year's paper, with correspondingly high thresholds for awards. The setting committee will be taking note of this when setting next year's paper.

This year we took the unusual step of giving specific guidance in B6 about what a complete solution looks like. Of course the guidance given applies equally to all problems, in all years, but as some students may not be familiar with what a solution looks like to a problem such as this we decided to make it more explicit. It was encouraging that many candidates did manage a mostly complete solution to B6, but it should also be noted that many candidates fell short in exactly the ways that the guidance was meant to address.

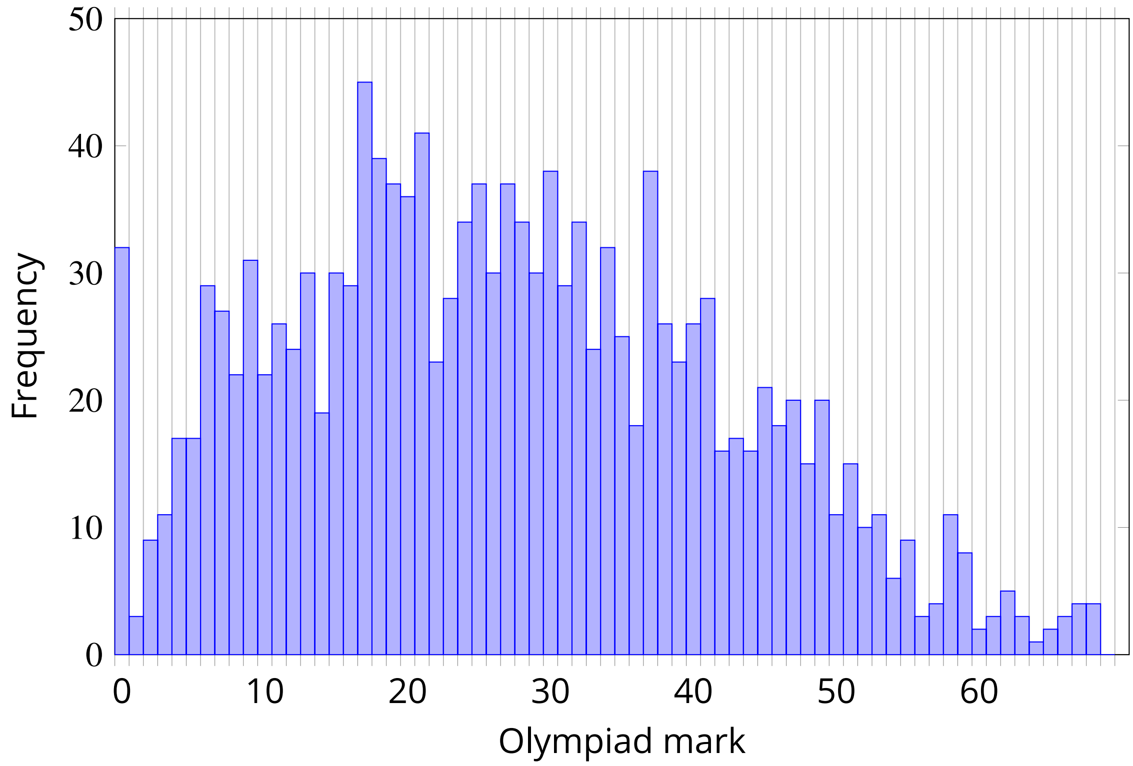
With this in mind, it seems worth mentioning once again that in all Olympiad papers solutions need to be completely justified. For many questions (including B6 this year) this means showing that all claimed solutions work, and that no other solutions can work. Showing that one particular proposed construction doesn't work in other cases is not sufficient. In other questions, which ask candidates to find the smallest solution, candidates must justify why no smaller solution could exist. The precise requirements will vary from question to question but the principle is the same: candidates must fully justify their solutions, against all requirements listed in the question.

The 2024 Junior Mathematical Olympiad attracted 1460 entries. The scripts were marked in London (with some remote markers) from the 21^{st} to the 23^{rd} of June by a team of: George Hugh Ainsley, Zack Bassman, Andrea Chlebikova, Daniel Claydon, Andy Cox, Juliette Culver, Laura Daniels, Wendy Dersley*, Ashling Dolan, David Dyer, Chris Eagle*, Ben Fairfax, Carol Gainlall, Chris Garton, Edward Godfrey, Anthony Goncharov, Amit Goyal, Aditya Gupta, Peter Hall*, Ben Handley, James Handy, Jon Hart, Alexander Hurst, Michael Illing, Thomas Kavanagh, Hayden Lam, Greta Leighton, Aleksandar Lishkov, Thomas Lowe, Sam Maltby*, Oliver Murray, David Phillips, Vivian Pinto, Jay Razdan, Heerpal Sahota, Niccolo Salvatori, Jacob Savage, Amit Shah, Gurjot Singh, Alan Slomson, Geoff Smith*, Matthew Smith, Amit Srivastava, Rob Summerson, Stephen Tate*, Tommy Walker Mackay, Emma Wheeler, Li Zhang. (An asterix shows that the marker was a problem captain).

The Section B problems were authored by, respectively, Howard Groves, Howard Groves, Unknown, Tony Gardiner, Alex Voice, and Unknown.

In addition to the written solutions in this report, video solutions can be found here.

Mark distribution



The mean score was 27 and the median score was 26.

The thresholds for medals, Distinction and Merit were as follows:

Gold Medal: 50+ (only Section B)

Silver Medal: 43+ (only Section B)

Bronze Medal: 38+ (only Section B)

Distinction: 38+ (A and B combined)

Merit: 20+ (A and B combined)

Question 1

What is the smallest positive integer that only contains the digits 0 and 1, and is divisible by 36?

Solution

Since $36 = 4 \times 9$, and since 4 and 9 have no factors in common, a number is a multiple of 36 if and only if it is a multiple of both 4 and 9. To be a multiple of 9, the digits must add up to a multiple of 9, which means that the number of 1s must be a multiple of 9. To be a multiple of 4, the last two digits must be a multiple of 4. Since none of 1, 10, nor 11 are multiples of 4, the only option is to put 0s in the last two digits. Since we need to use at least 9 1s and 2 0s, we need at least 11 digits. The solution 11111111100 is the only such 11 digit number, as we need the two 0s to go at the end. Any other solution would have more digits, and would therefore be larger.

Markers' comments

Being question 1, the hope was for this to be the most accessible question, but in fact fewer than half of students produced a mostly correct solution. Several common issue held students back from scoring highly:

Students often misquoted divisibility rule for 9 saying the digits must sum to 9. They could have said sum to a multiple of 9, or that the digital root is 9. Students who said this, could not score very highly, since it only leaves one possibility of 9 1's, rather than arguing why their answer must be the smallest.

Many students didn't argue why their answer was the smallest. The idea of adding 0's would make the number longer and the only required 0's were should be placed at the end to ensure the smallest number was all that was needed.

A lot of students didn't quite argue the divisibility for 4 correctly and often resorted to just adding 0's at the end. They said it must be even since 4 is even so it is 111 111 111 0 and said this doesn't work, the next is 111 111 111 00 but this isn't the next, the next would be 101 111 111 10 - hence were not correctly arguing why it is the smallest multiple of 36.

Many tried to consider multiples of 36 that end in 0 which lead to multiples of 180 which in turn lead to multiples of 900, but then found it more difficult to produce a robust argument.

Many students would miss out logical steps, i.e. stating that the number must be divisible by 4, therefore must end in 00 - but why? Or stating that the digits add to a multiple of 9 therefore there must be 9 1's. Without referring to why they have chosen this and why it allows it to be the smallest they have not fully answered the question.

The relationship between being divisible by 36 vs being divisible by both 4 and 9 is a little subtle and this cost some students a mark. It is clearly true that a multiple of 36 must be a multiple of 9 and a multiple of 4. It is slightly less obvious that a multiple of 4 and 9 must be a multiple of 36 (although this turns out to be true). Notice, for instance, that it wouldn't be true for multiples of 12 and 3, and in fact the number 11100 is a multiple of both 12 and 3. As such, to score full marks a student had to justify why their number is in fact a multiple of 36. Explicit division would be enough to prove it, but a simpler way is to mention that 4 and 9 have no factors in common. There is no need to use more advanced language such as 'coprime' at JMO level. While this penalty may seem harsh (and some students may have thought that the claim is obvious), is is important to always prove both directions of the required result; here, that means that anything smaller cannot be a multiple of 36, and that the given number is a multiple of 36. Remembering to always explicitly prove both directions is vital for all future Olympiads, and beyond.

Question 2

Natasha and Rosie are running at constant speeds in opposite directions around a running track. Natasha takes 70 seconds to complete each lap of the track and meets Rosie every 42 seconds.

How long does it take Rosie to complete each lap?

Solution

Let t be the length of the track, in m, and n and r be Natasha's and Rosie's running speeds, respectively, in m/s. From one meeting to the next, they need to have run a total distance of t between the two of them. So we have

$$\begin{aligned} t &= 70n \\ t &= 42(n+r) \\ \implies 42r &= 28n \end{aligned}$$

Rosie's lap time is $\frac{t}{r} = \frac{t}{n}\frac{n}{r} = 70 \times \frac{42}{28} = 105$s.

An alternative method is to divide up the track into 210 equal sections. It's tempting to assume that it's 210m long, but that's making an assumption beyond what the question tells us so we can't call these sections metres, instead let's call them squigs. Natasha's running speed is 3 squigs/second. The combined running speed of the two of them is 5 squigs/second. Therefore Rosie's speed is 2 squigs/second, so she takes $\frac{210}{2} = 105$ seconds to run a lap. Note that the size of a squig turned out not to matter – but that doesn't mean that we could assume that without justification.

Markers' comments

This question was very popular and very well answered, with almost half of those who attempted it getting full marks.

Good answers combined numerical calculation with a clear explanation of the meanings of the numbers being calculated. In particular, lack of clarity over whether a quantity was a time, a distance or a speed was a common way to lose marks.

A key question for students was how to measure distances, as the length of the track wasn't specified in the question. Since the track length was unspecified in the question, answers had to work for any possible length to achieve a high score.

Markers saw a variety of approaches:

- Treat the length of the track as "1", i.e. work in terms of fractions of a lap. This was the most common, and the most successful, approach
- Declare the track to be of some convenient length, like 70m or 700m. This was quite common and lost a lot of marks. Markers were slightly more generous to students who assumed a standard 400m running track.

- Call the track length x (or similar). The resulting algebra proved a challenge for some, but many good answers were seen.
- Express the track length in arbitrary units, as in the "squigs" approach in the official solutions. This was rarely seen but led to some very neat solutions.

Successful attempts tended to focus narrowly on two successive meetings of Natasha and Rosie and the distances the two had travelled between them. Those students who looked at three or more meetings, over multiple laps of the track, tended to struggle with the additional complexity.

Question 3

The positive integers from 1 to n ($n \geq 2$) inclusive are to be spaced equally around the
circumference of a circle so that:

(a) no two even numbers are adjacent;

(b) no two odd numbers are adjacent;

(c) no two numbers differing by 1 are adjacent.

What is the smallest value of n for which the above is possible?

Solution

$n = 2$ is clearly impossible as 1 and 2 would be next to each other. Therefore the digit 3 must appear. Since the neighbours of 3 must be even, and can't be 2 or 4, the smallest they can be is 6 and 8. This shows that we need $n \geq 8$. All that remains is to check that $n = 8$ is possible. The sequence, going around the circle, could be 1, 6, 3, 8, 5, 2, 7, 4, before coming back to the original 1, which satisfies all the conditions.

Markers' comments

This question was attempted by the majority of the students, with roughly one fifth achieving success. The key barriers to achievement in this question were: not realising the necessity to indicate why $n = 8$ is minimal; and failure to supply a clear explanation that it is not possible for $n < 8$.

This question required the student to show two key points:

(1) Show that there is an example for $n = 8$.

(2) Explain why it is not possible to do for $n < 8$.

The majority of students were able to give a successful example for $n = 8$, but, unfortunately, some students explained point (2) well and completely neglected point (1).

Some students took a constructive approach to finding the solution, but failed to appreciate whether their construction would necessarily give a minimal example. Choosing the smallest integers as neighbours locally does not guarantee that the whole collection of integers would be minimal.

Successful arguments made by students when considering whether $n = 8$ were minimal, included:

(i) Explaining why n must be even and completely covering the cases for $n = 2, 4$ and 6.

(ii) Considering neighbours of specific numbers that force lower bounds for n.

(iii) Considering the available numbers that can be adjacent to a specific number and counting these.

The cleanest solution to this question is:

i Explain why $n = 2$ is not possible.

ii We then know that the number 3 must appear in any arrangement and that each integer would have two neighbours. The smallest permissible neighbours of 3 are 6 and 8, meaning $n \geq 8$.

iii Give the example where $n = 8$, concluding that this is the minimal example.

The first step was often neglected and some students failed to appreciate that this fact should be explained or, at the least, mentioned. The best solutions gave the appropriate reason for ruling out $n = 2$.

If a student attempted to eliminate examples for values of $n < 8$, they would often miss crucial examples or not explain what assumptions they could make and why. Clear solutions that used this approach often alluded to the integers needing to alternate between odd and even numbers around the circle, which significantly reduced and simplified the diagrams to consider.

Some students provided an interesting argument that considered the number of possible neighbours for an (arbitrary) integer around the circle. These arguments often lacked some detail such as not considering 1 that has only one consecutive positive integer, or considering why each integer must have two neighbours. The argument made the observation that from even n integers, an integer m would have $\frac{n}{2}$ integers of the opposite parity, two of which would be consecutive to m i.e. $m - 1$ and $m + 1$. Each integer needs two neighbours, so the number of integers that are permitted to go next to the integer m would need to be at least 2, giving the inequality: $\frac{n}{2} - 2 \geq 2$, which simplifies to $n \geq 8$. This only works when $1 < m < n$ and once we deduce $n > 2$, which was sometimes lacking.

Some students also considered arithmetic sequences with common difference 3 after realising that 3 is the minimal difference between any two neighbours around the circle. Although this idea was helpful when constructing a solution for $n = 8$, the argument why it would lead to a minimal solution was usually not complete.

The key learning points for students who tackled this question would be to consider carefully all assumptions and steps in their arguments.

Question 4

My piggy bank contains x pound coins and y pennies and rattles nicely. If instead it contained y pound coins and x pennies, then I would only have half as much money.

What is the smallest amount of money my piggy bank could contain?

Solution

The wording here is a bit curious, with the reference to rattling. While it is easy to dismiss it as whimsy, the important conclusion from the rattling is that at least one of x and y is non-zero. The next sentence tells us that

$$100x + y = 2(100y + x)$$

ie $98x = 199y$. The solution $x = y = 0$ is contradicted by the rattling, so we need to find the smallest positive solution. Since the RHS is a multiple of 199, so must the LHS be. Since 98 has no prime factors in common with 199 (199 is, in fact, prime) we know that x must be a multiple of 199. So the smallest solution is $x = 199$, $y = 98$, for a total amount of money of £199.98.

Markers' comments

This question was generally answered well and the majority of students found the correct solution. A key factor in the best answers was an algebraic statement for the value of the money in the piggy bank. Students who wrote the value as $100x + y$ were the most likely to write a similar statement for the situation where the number of coins were reversed.

Such solutions quickly arrived at $199y = 98x$ and then commented about lack of common factors to find the correct solution. We expected a comment to illustrate why $x = 199$ and $x = 98$ was the lowest possible solution. Many students used either highest common factor or lowest common multiple to explain this.

Only rarely did students refer to the "rattles nicely" and explain why zero was not a possible solution.

A small minority of students did not appear to understand pounds and pence, some solutions used other ratios (1:10 and 1:12 were seen)

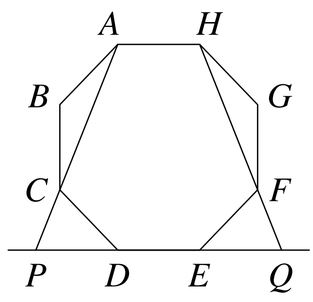
We did not find any students who arrived at the correct solution by other means. Those who attempted a "trial and error" approach were overwhelmed and made no progress.

There were other students who rather overused algebraic approaches - some using a different letter for the pound coins in each case, or writing statements for the number of coins - rather than the value of coins. Students are encouraged to use algebra at this level, but they need to ensure the algebra is relevant and useful.

Question 5

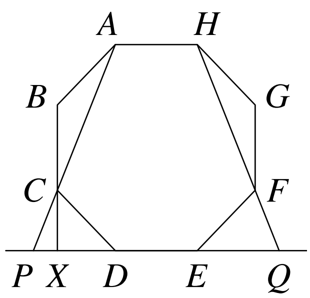
A regular octagon $ABCDEFGH$ has sides of length 1. The lines AC and HF meet the line going through D and E at P and Q respectively.

What is the length of the line PQ?



Solution

Extend line BC until it meets line PQ at point X.



The exterior angles of a regular octagon are $45°$ so the interior angles are $135°$. So $\angle ABC = 135°$.

Triangle ABC is isosceles, as two of its sides are sides of the regular octagon. Therefore we have $\angle BAC = \angle BCA$. Since angles in a triangle add to $180°$, these two angles add to $45°$ and so we have $\angle BAC = \angle BCA = 22.5°$. Since $\angle BCA = \angle PCX$ (vertically opposite), we have $\angle PCX = 22.5°$.

$\angle XCD = 45°$ (exterior angle of the octagon), so $\angle PCD = \angle PCX + \angle XCD = 67.5°$. $\angle CDP = 45°$ (exterior angle) and angles in triangle PCD add to $180°$ so $\angle CPD = 67.5°$. Thus triangle CPD is in fact isosceles, and so $PD = CD = 1$.

The same argument holds on the other side, giving $PQ = 3$.

Markers' comments

This question concerned a property of lines associated with a regular octagon. At this stage of their education, many JMO candidates know only a limited amount of Euclidean geometry. As we expected, the majority of successful solutions involved naïve angle chases and observations about isosceles triangles. A slightly more sophisticated approach is available via the result that opposite sides of a parallelogram have equal length, and some contestants found this. A remark about isosceles triangles: the marking team found over 25 different spellings of the word iskoquelees among the scripts.

A metric approach by calculation was also possible, and some contestants used the theorem of Pythagoras and a similarity argument to great effect. The marking team took the approach that candidates could use 'obvious' facts about regular octagons without proof; there are families of parallel diagonals and sides, and some families of parallel diagonals are perpendicular to other families of parallel diagonals. We allowed arguments based on the symmetry of the figure.

A few contestants who approached this problem by calculating angles struggled to divide by 2 and to subtract two numbers. Confident and accurate elementary arithmetic (without calculators) is an essential skill.

The official solutions, prepared ahead of marking, include some advanced methods of proof. These are for the amusement of teachers and some precocious students.

Question 6

Let A be the set of $2n$ positive integers $1, 2, 3, \dots, 2n$, where $n \geq 1$.
For which values of n can this be split into n pairs of integers in such a way that every pair has a sum which is a multiple of 3?
As always in Olympiad problems such as this, you also need to explain why no other values of n are possible.

Solution

Suppose that n is 1 more than a multiple of three. Then $2n$ is one less than a multiple of three, and we have an easy solution: pair the first with the last, the second with the second last, etc. So in this case we can definitely achieve the required splitting.

Suppose n is a multiple of 3, so $n = 3k$. Then the last four integers are $6k - 3, 6k - 2, 6k - 1, 6k$. We can pair these up as $(6k - 3, 6k)$ and $(6k - 2, 6k - 1)$. This leaves a smaller problem, with $n' = n - 2$ taking the role of n in the original problem. Since n' is one more than a multiple of three, these remaining integers can also be split in the required way.

Now suppose that n is 1 less than a multiple of 3. Consider the sum $1 + 2 + \cdots + 2n$. Since $n+1$ is a multiple of 3, we can split the set $\{1, 2, \dots, 2(n+1)\}$ into pairs whose sum is a multiple of 3, as shown above. So $1 + 2 + \cdots + 2(n + 1)$ is a multiple of 3. But $1 + 2 + \cdots + 2(n + 1) = (1 + 2 + \cdots + 2n) + 4n + 3$, and $4n + 3$ is not divisible by 3 as n is not divisible by 3. So $1 + 2 + \cdots + 2n$ is not divisible by 3, and it is impossible to achieve the required split.

Alternatively, starting from the same sum $1 + 2 + \cdots + 2n$, we can pair the terms up from the outside in: $1 + 2n$, $2 + (2n - 1)$, etc, each of which sum to $2n + 1$. There are n such pairs, so the total sum is $n(2n + 1)$. Since neither n nor $2n + 1$ is a multiple of 3 in this case, the product is also not a multiple of 3 (note that this argument relies on 3 being prime).

So in summary, any value of n which is either a multiple of 3, or 1 more than a multiple of 3, can be split in the required way, and no others.

Markers' comments

This question was attempted by about half the students, with roughly half of those making some progress towards a solution, which was good to see as it involved some advanced ideas for this age group. Relatively few scored highly as it needed some care for students to keep all their ideas in order and be clear about what they were trying to accomplish at each stage.

There were quite a few ways that arguments proceeded, either for how to construct pairings or proving that pairings did not exist. Some students have seen modular arithmetic or mathematical induction and so were able to write in fairly sophisticated ways, but others not familiar with these concepts still found lucid ways to explain the core ideas and did not appear to be in any way disadvantaged. Indeed, in some cases students who had clearly seen some more advanced ideas were too keen to demonstrate their knowledge when a simpler route was available.

The two main pitfalls were of a similar nature, namely thinking that if a particularly argument that worked in one case failed in another, then the second case necessarily had the opposite outcome to the first case. This is exactly the pitfall that the italicised note in the question was meant to warn against.

Students would give an algorithm for constructing pairings for when n was $3a$ or $3a + 1$, but they would then assume that since that particular algorithm didn't extend to $3a + 2$, no pairing was possible. (Likewise, many students paired 1 with $2n$, 2 with $2n - 1$, 3 with $2n - 2$, etc. to prove any $n = 3a + 1$ produced a set of pairings and then said that since this way of pairing didn't work for $n = 3a$, then no solution existed for n being a multiple of 3.) Since the impossibility of $n = 3a + 2$ hasn't been shown, such scripts could not score highly.

Conversely, many proved that if $n = 3a + 2$ then A could not be split into pairs, but then said that if $n = 3a$ or $n = 3a + 1$ then this wasn't a problem and therefore a pairing did exist. Since these scripts haven't shown that $n = 3a$ and $n = 3a + 1$ are possible, they also can't score highly. One common argument was that the sum of the elements of A was $n(2n + 1)$, and this had to be a multiple of 3 (as it's also the sum of the sums of the pairs), and if $n \equiv 2 \pmod 3$ then $n(2n + 1) \equiv 1 \pmod 3$. They then said that when n was 0 or 1 mod 3 then $n(2n + 1)$ was a multiple of 3, so n would necessarily work since the sum of the elements of A was therefore a multiple of 3. As a counterexample, the set $\{2, 5, 8, 9\}$ has a sum of its elements being 24, yet it is not possible to split it into pairs which sum to multiples of 3.