IYGB – MP2 PAPER X – QUESTION 1

WITHOUT LOSS OF GENERALITY LET THE A, B & C BE

$A = A \neq 0$

$B = Ar$

$C = Ar^2$ (AS THEY ARE IN GEOMETRIC PROGRESSION)

NOW A, 2B & C ARE IN ARITHMETIC PROGRESSION

$\Longrightarrow 2B - A = C - 2B$

$\Longrightarrow 4B = A + C$

$\Longrightarrow 4Ar = A + Ar^2$

$\Longrightarrow 4r = 1 + r^2 \quad (A \neq 0)$

$\Longrightarrow r^2 - 4r + 1 = 0$

$\Longrightarrow (r-2)^2 - 4 + 1 = 0$

$\Longrightarrow (r-2)^2 = 3$

$\Longrightarrow r - 2 = \begin{cases} \sqrt{3} \\ -\sqrt{3} \end{cases}$

$\Longrightarrow r = \begin{cases} 2+\sqrt{3} \\ 2-\sqrt{3} \end{cases}$

MP2 – PAPER X – QUESTION 2

a) SOLVING SIMULTANEOUSLY WE OBTAIN

$$\left.\begin{array}{l} y = \cos 2x \\ y = \sin x \end{array}\right\} \Longrightarrow \quad \cos 2x = \sin x$$

$$1 - 2\sin^2 x = \sin x$$

$$0 = 2\sin^2 x + \sin x - 1$$

$$0 = (2\sin x - 1)(\sin x + 1)$$

$$\sin x = \begin{cases} 1/2 \\ -1 \end{cases}$$

- $x = \pi/6 \pm 2n\pi$
- $x = \frac{5\pi}{6} \pm 2n\pi$

$n = 0, 1, 2, 3, \ldots$

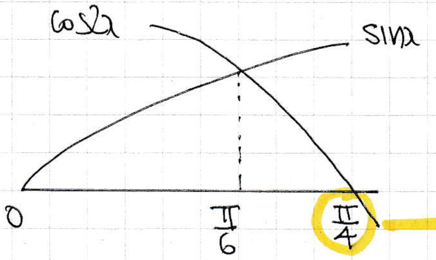
OR

- $x = -\pi/2 \pm 2n\pi$
- $x = 3\pi/2 \pm 2n\pi$

$n = 0, 1, 2, 3, 4, \ldots$

$\therefore$ THE FIRST POSITIVE SOLUTION IS $x = \pi/6$

b) LOOKING AT THE DIAGRAM WE HAVE



$\cos 2x = 0$

$2x = \pi/2 \pm 2n\pi$

$2x = 3\pi/2 \pm 2n\pi$

$n = 0, 1, 2, 3, \ldots$

$x = \frac{\pi}{4} \pm n\pi$

$x = \frac{3\pi}{4} \pm n\pi$

$$\text{AREA} = \int_0^{\pi/6} \sin x \, dx + \int_{\pi/6}^{\pi/4} \cos 2x \, dx$$

$$= \Big[-\cos x\Big]_0^{\pi/6} + \Big[\tfrac{1}{2}\sin 2x\Big]_{\pi/6}^{\pi/4}$$

$$= \left[-\frac{\sqrt{3}}{2} - (-1)\right] + \left[\frac{1}{2} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right] = -\frac{\sqrt{3}}{2} + 1 + \frac{1}{2} - \frac{\sqrt{3}}{4} = \frac{3}{2} - \frac{3}{4}\sqrt{3}$$

$$\frac{3}{4}(2 - \sqrt{3})$$

IYGB - MP2 PAPER X - QUESTION 3

START BY REGROUPING THE TERMS

$$\Rightarrow x - 2x^2 + x^3 - 2x^4 + x^5 - 2x^6 + \dots = -\frac{2}{5}$$

$$\Rightarrow \left(x + x^3 + x^5 + \dots\right) - 2\left(x^2 + x^4 + x^6 + \dots\right) = -\frac{2}{5}$$

G.P WITH $a = x$, $r = x^2$

$$S_\infty = \frac{x}{1-x^2}$$

G.P. WITH $a = x^2$, $r = x^2$

$$S_\infty = \frac{x^2}{1-x^2}$$

HENCE WE NOW HAVE

$$\Rightarrow \frac{x}{1-x^2} - 2\left(\frac{x^2}{1-x^2}\right) = -\frac{2}{5}$$

$$\Rightarrow \frac{x}{1-x^2} - \frac{2x^2}{1-x^2} = -\frac{2}{5}$$

$$\Rightarrow \frac{x-2x^2}{1-x^2} = \frac{-2}{5}$$

$$\Rightarrow 5x - 10x^2 = -2 + 2x^2$$

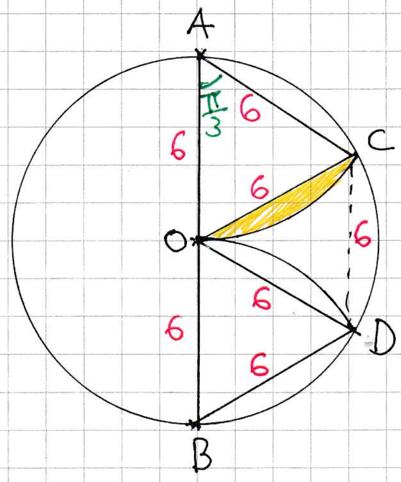
$$\Rightarrow 0 = 12x^2 - 5x - 2$$

$$\Rightarrow 0 = (4x+1)(3x-2)$$

$$\Rightarrow x = \begin{cases} \frac{2}{3} \\ -\frac{1}{4} \end{cases}$$

IYGB - MP2 PAPER X - QUESTION 4

LOOKING AT THE DIAGRAM, $\overset{\Delta}{AOC}$ IS EQUILATERAL, SO ALL ITS ANGLES ARE $60° = \frac{\pi}{3}$



- AREA OF SECTOR $O\overset{\frown}{A}C$

$$= \frac{1}{2}r^2\theta^c$$
$$= \frac{1}{2} \times 6^2 \times \frac{\pi}{3}$$
$$= 6\pi$$

- AREA OF EQUILATERAL TRIANGLE $\overset{\Delta}{AOC}$

$$= \frac{1}{2}|AO||AC|\sin\frac{\pi}{3}$$
$$= \frac{1}{2} \times 6 \times 6 \times \frac{\sqrt{3}}{2}$$
$$= 9\sqrt{3}$$

- AREA OF SEGMENT, SHOWN IN YELLOW

$$= 6\pi - 9\sqrt{3}$$

- REQUIRED AREA = AREA OF SECTOR $C\overset{\frown}{O}D$ − 2 "YELLOW" SEGMENTS

$$= 6\pi - 2(6\pi - 9\sqrt{3})$$
$$= 6\pi - 12\pi + 18\sqrt{3}$$
$$= 18\sqrt{3} - 6\pi$$
$$= 6(3\sqrt{3} - \pi)$$

// AS REQUIRED

IYGB - MP2 PAPER X - QUESTION 5

a) PROCEED AS FOLLOWS NOTING $-1 \leqslant x \leqslant 1$ & $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$

$$y = \arcsin x$$
$$\sin y = x$$
$$x = \sin y$$
$$\frac{dx}{dy} = \cos y$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$
$$\frac{dy}{dx} = \frac{1}{+\sqrt{1-\sin^2 y}}$$

AS $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$
$0 \leqslant \cos y \leqslant 1$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

b) $y = 2\arcsin x - 4x^{\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}} - 6x^{\frac{1}{2}}$$

FOR STATIONARY VALUES

$$\Rightarrow 0 = \frac{2}{(1-x^2)^{\frac{1}{2}}} - 6x^{\frac{1}{2}}$$
$$\Rightarrow 6x^{\frac{1}{2}} = \frac{2}{(1-x^2)^{\frac{1}{2}}}$$
$$\Rightarrow 3x^{\frac{1}{2}} = \frac{1}{(1-x^2)^{\frac{1}{2}}}$$
$$\Rightarrow 9x = \frac{1}{1-x^2}$$

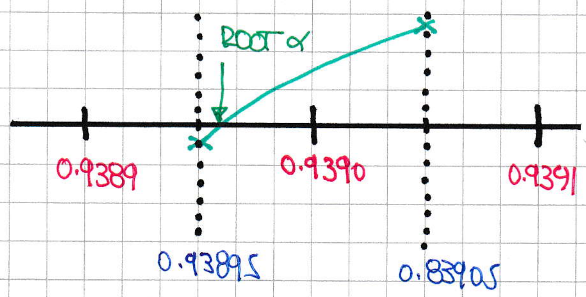
IYGB - MP2 PAPER X - QUESTION 5

$$\Longrightarrow \quad 9x - 9x^3 = 1$$

$$\Longrightarrow \quad 9x^3 - 9x + 1 = 0$$

c) LET $f(x) = 9x^3 - 9x + 1$

- $f(0.93905) = 0.00116\ldots > 0$
- $f(0.93895) = -0.00031\ldots < 0$



AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN IN THE ABOVE INTERVAL

$$0.93895 < \alpha < 0.93905$$

$\therefore \quad \alpha = 0.9390$ CORRECT TO 4 d.p.

d) ASSUMING NO KNOWLEDGE OF "$\alpha\beta\gamma = -\frac{d}{a}$"

$$\Longrightarrow \quad 9x^3 - 9x + 1 = 0$$

$$\Longrightarrow \quad x^3 - x + \frac{1}{9} = 0$$

$$\Longrightarrow \quad (x - 0.9390)(x + 1.0515)(x - \gamma) = 0$$

THUS $\quad (-0.9390)(1.0515)(-\gamma) = \frac{1}{9}$

$$0.9873585\ldots\gamma = \frac{1}{9}$$

$$\gamma = 0.112$$

3 d.p

IYGB - MP2 PAPER X - QUESTION 6

FORMING A DIFFERENTIAL EQUATION

$$\frac{dm}{dt} = -km$$

RATE

DECAYING

PROPORTIONAL

MASS OF ISOTOPE PRESENT

m = MASS (SAY IN kg)

t = TIME (YEARS)

$t=0 \quad m=M$

$t=80 \quad m=\frac{1}{2}M$

SEPARATING VARIABLES

$\Rightarrow dm = -km\,dt$

$\Rightarrow \frac{1}{m}\,dm = -k\,dt$

$\Rightarrow \int \frac{1}{m}\,dm = \int -k\,dt$

$\Rightarrow \ln|m| = -kt + C$

$\Rightarrow m = e^{-kt+C}$

$\Rightarrow m = e^{-kt} \times e^{C}$

$\Rightarrow m = Ae^{-kt} \qquad (A = e^{C})$

APPLY CONDITION $m = M$, AT $t=0$

$\Rightarrow M = Ae^{0}$

$\Rightarrow A = M$

IYGB – MP2 PAPER X – QUESTION 6

$$\Longrightarrow \quad m = Me^{-kt}$$

"HALF LIFE OF 80" $\Longrightarrow$ $t = 80 \quad m = \frac{1}{2}M$

$$\Longrightarrow \quad \frac{1}{2}M = Me^{-80k}$$

$$\Longrightarrow \quad \frac{1}{2} = e^{-80k}$$

$$\Longrightarrow \quad 2 = e^{80k}$$

$$\Longrightarrow \quad \left(e^{10k}\right)^{8} = 2$$

$$\Longrightarrow \quad e^{10k} = 2^{\frac{1}{8}} \approx 1.0905\ldots \qquad (\text{OR } k \simeq 0.0086643\ldots)$$

FINALLY WHEN $t = 50$

$$\Longrightarrow \quad m = Me^{-kt}$$

$$\Longrightarrow \quad m = Me^{-50k}$$

$$\Longrightarrow \quad m = M\left(e^{10k}\right)^{-5}$$

$$\Longrightarrow \quad m = M \times 2^{-\frac{5}{8}}$$

$\therefore$ PROPORTION WHICH REMAINS $= \dfrac{2^{-\frac{5}{8}}\cancel{M}}{\cancel{M}} = \dfrac{1}{2^{\frac{5}{8}}} \simeq 0.648$

$\simeq 64.8\%$

IYGB – MP2 PAPER X – QUESTION 7

DIFFERENTIATE WITH RESPECT TO x, TWICE

$$y = \ln(1+\sin x)$$

$$\frac{dy}{dx} = \frac{\cos x}{1+\sin x}$$

$$\frac{d^2y}{dx^2} = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\sin x - 1}{(1+\sin x)^2} = -\frac{1+\sin x}{(1+\sin x)^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{1+\sin x}$$

BUT SINCE $y = \ln(1+\sin x) \implies 1+\sin x = e^y$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{e^y}$$

$$\frac{d^2y}{dx^2} = -e^{-y}$$

i.e. $f(y) = -e^{-y}$

IYGB – MP2 PAPER X – QUESTION 8

USING THE DOUBLE ANGLE IDENTITY FOR SINE

$$\Longrightarrow \sin x \cos x \cos 2x \cos 4x = \frac{\sqrt{2}}{16} \quad \big\downarrow \times 2$$

$$\Longrightarrow \underbrace{2\sin x \cos x}\cos 2x \cos 4x = \frac{\sqrt{2}}{8}$$

$$\Longrightarrow \sin 2x \cos 2x \cos 4x = \frac{\sqrt{2}}{8} \quad \big\downarrow \times 2$$

$$\Longrightarrow \underbrace{2\sin 2x \cos 2x}\cos 4x = \frac{\sqrt{2}}{4}$$

$$\Longrightarrow \sin 4x \cos 4x = \frac{\sqrt{2}}{4} \quad \big\downarrow \times 2$$

$$\Longrightarrow \underbrace{2\sin 4x \cos 4x} = \frac{\sqrt{2}}{2}$$

$$\Longrightarrow \sin 8x = \frac{\sqrt{2}}{2}$$

$$\left(\begin{array}{l} 8x = \pi/4 \pm 2n\pi \\ 8x = 3\pi/4 \pm 2n\pi \end{array} \right. \qquad n = 0, 1, 2, 3, \ldots$$

$$\left(\begin{array}{l} x = \pi/32 \pm \frac{n\pi}{4} \\ x = 3\pi/32 \pm n\pi/4 \end{array} \right.$$

FOR THE RANGE $0 \leq x \leq \pi/2$

$$x = \frac{\pi}{32}, \frac{9\pi}{32}, \frac{3\pi}{32}, \frac{11\pi}{32}$$

IYGB – MP2 PAPER X – QUESTION 9

a) PROCEED BY PARTIAL FRACTION OR DIRECT EVALUATION

$$\Rightarrow f(x) = a(2-3x)(1-2x)^{-1}(2+x)^{-1}$$

$$\Rightarrow f(x) = a(2-3x)(1-2x)^{-1} \times 2^{-1}\left(1+\tfrac{1}{2}x\right)^{-1}$$

$$\Rightarrow f(x) = \tfrac{1}{2}a(2-3x)(1-2x)^{-1}\left(1+\tfrac{1}{2}x\right)^{-1}$$

$$\Rightarrow f(x) = \tfrac{1}{2}a(2-3x)\left[1+(-1)(-2x)+\frac{(-1)(-2)}{1\times 2}(-2x)^2+\dots\right]\left[1+(-1)\left(\tfrac{1}{2}x\right)+\frac{(-1)(-2)}{1\times 2}\left(\tfrac{1}{2}x\right)^2+\dots\right]$$

$$\Rightarrow f(x) = \tfrac{1}{2}a(2-3x)\left(1+2x+4x^2+\dots\right)\left(1-\tfrac{1}{2}x+\tfrac{1}{4}x^2+\dots\right)$$

EXPAND UP TO x^2 DUE TO PART (b)

$$\Rightarrow f(x) = \tfrac{1}{2}a(2-3x)\begin{bmatrix} 1 - \tfrac{1}{2}x + \tfrac{1}{4}x^2 + \dots \\ 2x - x^2 + \dots \\ + 4x^2 + \dots \end{bmatrix}$$

$$\Rightarrow f(x) = \tfrac{1}{2}a(2-3x)\left(1+\tfrac{3}{2}x+\tfrac{13}{4}x^2\dots\right)$$

$$\Rightarrow f(x) = \tfrac{1}{2}a\begin{bmatrix} 2 + 3x + \tfrac{13}{2}x^2 + \dots \\ -3x - \tfrac{9}{2}x^2 + \dots \end{bmatrix}$$

$$\Rightarrow f(x) = \tfrac{1}{2}a\left(2+2x^2+\dots\right)$$

$\therefore$ COEFFICIENT OF x IS ZERO

b) FINALLY FROM THE ABOVE EXPRESSION

$$\tfrac{1}{2}a(\dots 2x^2) = 10x^2$$

$$a = 10$$

IYGB - MP2 PAPER X - QUESTION 10

a) DIFFERENTIATING PARAMETRICALLY

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{2t} = \frac{1}{t}$$

EQUATION OF TANGENT AT $P(p^2, 2p)$, GRADIENT $\frac{1}{p}$

$$y - 2p = \frac{1}{p}(x - p^2)$$

$$0 - 2p = \frac{1}{p}(x - p^2)$$

$$-2p^2 = x - p^2$$

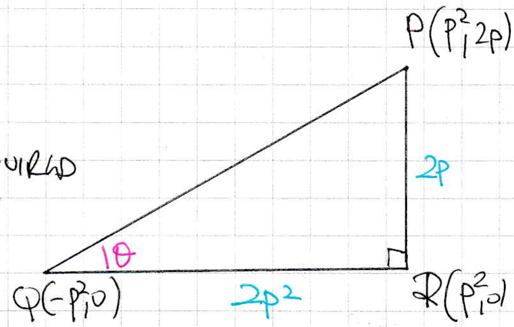
$$-p^2 = x$$

$\therefore Q(-p^2, 0)$

b) LOOKING AT THE TRIANGLE $P\hat{Q}R$

$$\tan\theta = \frac{|PR|}{|QR|} = \frac{2p}{2p^2} = \frac{1}{p}$$

AS REQUIRED



c) LOOKING AT THE TRIANGLE $P\hat{S}R$

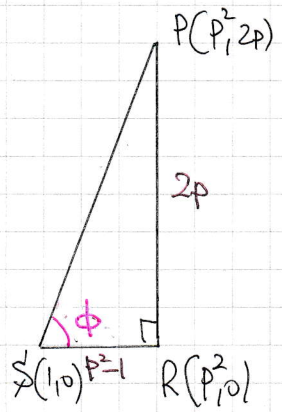
FROM DIAGRAM $\tan\phi = \frac{2p}{p^2 - 1}$

$\tan\theta = \frac{1}{p}$

$$\tan\phi = \frac{2\left(\frac{1}{\tan\theta}\right)}{\left(\frac{1}{\tan\theta}\right)^2 - 1}$$

$$\tan\phi = \frac{\frac{2}{\tan\theta}}{\frac{1}{\tan^2\theta} - 1}$$

$$\tan\phi = \frac{\frac{2}{\tan\theta} \times \tan^2\theta}{\frac{1}{\tan^2\theta} \times \tan^2\theta - 1 \times \tan^2\theta}$$



IYGB – MP2 PAPER X – QUESTION 10

$$\tan\phi = \frac{2\tan\theta}{1-\tan^2\theta}$$

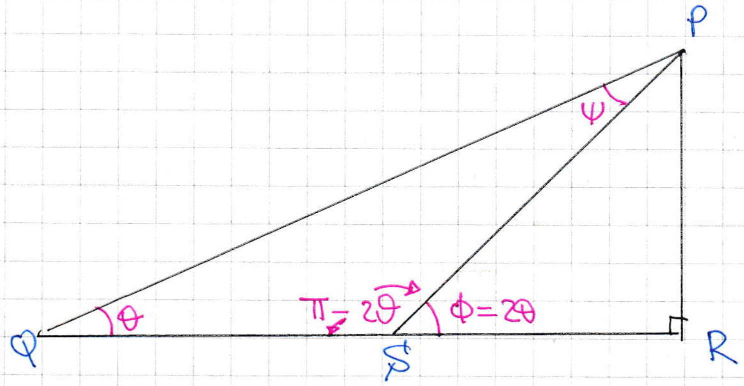
$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

$$\tan\phi = \tan(2\theta)$$

$$\phi = 2\theta$$

AS REQUIRED

d) LOOKING AT THE DIAGRAM BELOW



$$\theta + (\pi - 2\theta) + \psi = \pi$$
$$-\theta + \psi = 0$$
$$\psi = \theta$$

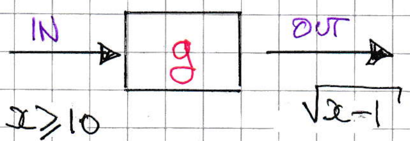
$\therefore$ PQS IS ISOSCELES $\Longrightarrow |QS| = |PS|$

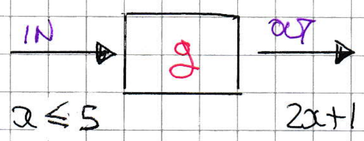
IYGB - MP2 PAPER X - QUESTION 11

a) STARTING WITH THE COMPOSITION

$$f(g(x)) = f\left(\sqrt{x-1}\right) = 2\sqrt{x-1}+1$$

NEXT THE DOMAIN





IT MUST SATISFY BOTH

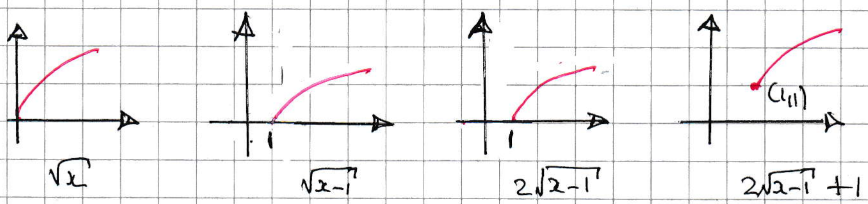
$x \geqslant 10$ AND $\sqrt{x-1} \leqslant 5$

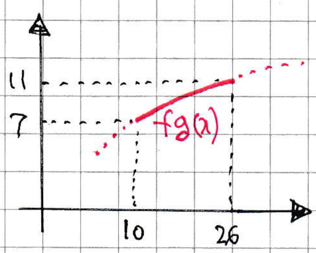
$x-1 \leqslant 25$

$x \leqslant 26$

$\therefore$ DOMAIN $x \in \mathbb{R}$ SUCH THAT $10 \leqslant x \leqslant 26$

TO FIND THE RANGE WE NEED TO SEE THE GRAPH OF $f(g(x))$



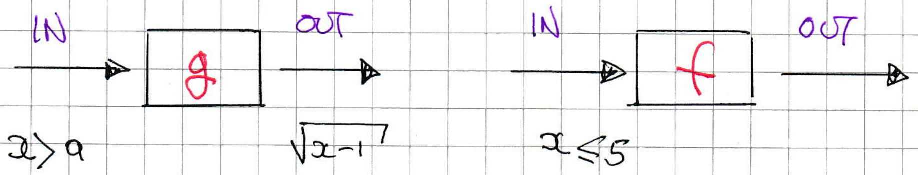


RANGE OF $f(g(x))$

$f(g(x)) \in \mathbb{R}$, SUCH THAT $7 \leqslant f(g(x)) \leqslant 11$

IYGB – MP2 PAPER X – QUESTION 11

b) LOOKING AGAIN AT THE DIAGRAM OF THE COMPOSITION



$$\sqrt{x-1} > 5$$

$$\sqrt{a-1} > 5$$

$$a - 1 > 26$$

$$a > 26$$

It $\quad a = 26$

IYGB - MP2 PAPER X - QUESTION 12

LOOKING AT THE "COMBINATION" $\sin x \sin 3x$ WE WORK AS FOLLOWS

$$\cos(3x+x) = \cos 3x \cos x - \sin 3x \sin x$$
$$\cos(3x-x) = \cos 3x \cos x + \sin 3x \sin x$$

SUBTRACTING "UPWARDS"

$$\cos(3x-x) - \cos(3x+x) = 2\sin 3x \sin x$$
$$\cos 2x - \cos 4x = 2\sin 3x \sin x$$

RETURNING TO THE INTEGRAL

$$\int_0^{\pi/3} 32\sin x \sin 2x \sin 3x \, dx = \int_0^{\frac{\pi}{3}} 16\sin 2x \left[2\sin 3x \sin x\right] dx.$$

$$= \int_0^{\frac{\pi}{3}} 16\sin 2x \left[\cos 2x - \cos 4x\right] dx = \int_0^{\frac{\pi}{3}} 16\sin 2x\cos 2x - 16\sin 2x \cos 4x \, dx.$$

$$= \int_0^{\frac{\pi}{3}} 8(2\sin 2x \cos 2x) - 16\sin 2x(2\cos^2 2x - 1) \, dx$$

$$= \int_0^{\frac{\pi}{3}} 8\sin 4x - 32\cos^2 2x \sin 2x + 16\sin 2x \, dx$$

$$= \left[-2\cos 4x + \frac{16}{3}\cos^3 2x - 8\cos 2x\right]_0^{\frac{\pi}{3}}$$

$$= \left[-2\left(-\tfrac{1}{2}\right) + \tfrac{16}{3}\left(-\tfrac{1}{2}\right)^3 - 8\left(-\tfrac{1}{2}\right)\right] - \left[-2 + \tfrac{16}{3} - 8\right]$$

$$= 1 - \frac{2}{3} + 4 + 2 - \frac{16}{3} + 8$$

$$= 9$$

IYGB - MP2 PAPER X - QUESTION 13

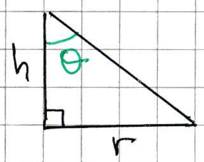
START BY CONNECTING DERIVATIVES

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times 50$$

$$\frac{dh}{dt} = \frac{1}{9\pi h^2} \times 50$$

$$\frac{dh}{dt} = \frac{50}{9\pi h^2}$$



$\tan\theta = 3$

$\frac{r}{h} = 3$

$r = 3h$

VOLUME OF CONE

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi (3h)^2 h$

$V = 3\pi h^3$

DIFFERENTIATE W.R.T h

$\frac{dV}{dh} = 9\pi h^2$

$\frac{dh}{dV} = \frac{1}{9\pi h^2}$

NEXT WE ARE TOLD THAT

... CONSTANT RATE OF 50 cm^3 PER SECOND...

... IN 60 SECONDS...

$V = 50 \times 60$

$V = 3000 \, cm^3$

CONVERT THIS VOLUME INTO h

$\Rightarrow V = 3\pi h^3$

$\Rightarrow 3000 = 3\pi h^3$

$\Rightarrow h^3 = \frac{1000}{\pi}$

$\Rightarrow h = \frac{10}{\pi^{1/3}}$

FINALLY WE HAVE

$$\Rightarrow \left.\frac{dh}{dt}\right|_{t=60} = \left.\frac{dh}{dt}\right|_{V=3000} = \left.\frac{dh}{dt}\right|_{h=\frac{10}{\pi^{1/3}}}$$

$$\Rightarrow \left.\frac{dh}{dt}\right|_{t=60} = \frac{50}{9\pi\left(\frac{10}{\pi^{1/3}}\right)^2} = \frac{50}{900\pi^{1/3}}$$

$$\Rightarrow \left.\frac{dh}{dt}\right|_{t=60} = \frac{1}{18\pi^{\frac{1}{3}}}$$

$$\Rightarrow \left.\frac{dh}{dt}\right|_{t=60} = \frac{1}{18}\pi^{-\frac{1}{3}}$$

AS REQUIRED