IYGB - MP2 PAPER I - QUESTION 1

BY THE QUOTIENT RULE

$$y = \frac{x^2}{\ln x} \implies \frac{dy}{dx} = \frac{\ln x \times 2x - x^2 \times \frac{1}{x}}{(\ln x)^2}$$

$$\implies \frac{dy}{dx} = \frac{2x\ln x - x}{(\ln x)^2}$$

$$\implies \left.\frac{dy}{dx}\right|_{x=\sqrt{e}} = \frac{2\sqrt{e}\ln\sqrt{e} - \sqrt{e}}{(\ln\sqrt{e})^2}$$

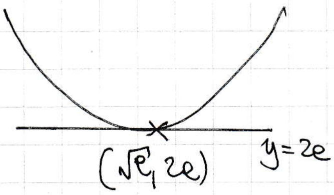
$$\implies \left.\frac{dy}{dx}\right|_{x=\sqrt{e}} = \frac{\sqrt{e} - \sqrt{e}}{\frac{1}{4}} = 0$$

IE STATIONARY POINT

FIND THE y CO-ORDINATE

$$y = \frac{(\sqrt{e})^2}{\ln\sqrt{e}} = \frac{e}{\frac{1}{2}} = 2e \qquad \text{IE} \quad (\sqrt{e}, 2e)$$

HENCE THE EQUATION OF THE TANGENT IS $y = 2e$



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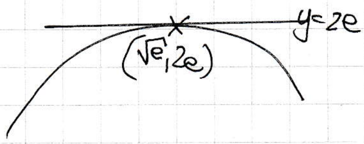


IYGB – MP2 PAPER I – QUESTION 2

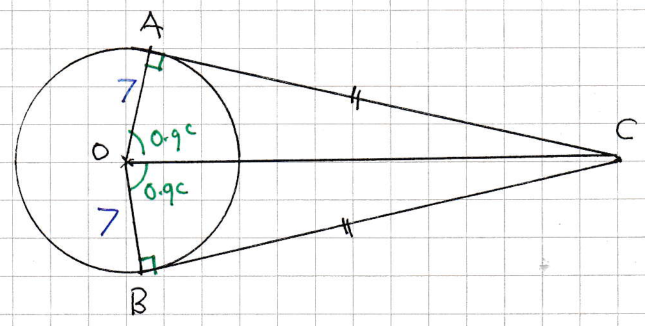
LOOKING AT THE DIAGRAM, ON THE RIGHT ANGLED TRIANGLE $\hat{AOC}$

$$\tan(0.9^c) = \frac{|AC|}{|AO|}$$

$$\tan(0.9^c) = \frac{|AC|}{7}$$

$$|AC| = 7\tan(0.9^c)$$

$$|AC| = 8.821107\ldots \text{ cm}$$



NOW THE AREA OF $\hat{AOC}$

$$= \frac{1}{2}|AO||AC| = \frac{1}{2} \times 7 \times 8.821\ldots = 30.87387\ldots \text{ cm}^2$$

AND THE AREA OF THE KITE AOBC IS EVIDENTLY TWICE THE AREA $\hat{AOC}$

$$\text{"KITE AOBC"} = 2 \times 30.87387\ldots = 61.74775\ldots \text{ cm}^2$$

NEXT THE AREA OF THE SECTOR

$$\text{"}\frac{1}{2}r^2\theta^c\text{"} = \frac{1}{2} \times 7^2 \times 1.8 = 44.1 \text{ cm}^2$$

FINALLY THE REQUIRED SHADED AREA IS GIVEN BY

$$61.74775\ldots - 44.1 = 17.64 \text{ cm}^2$$

IYGB – MP2 PAPER I – QUESTION 3

a) OBTAIN THE LIMITS FIRST

- AT A, $x=0$

$t^2=0$

$t=0$

- AT B, $y=0$

$1+\cos t=0$

$\cos t=-1$

$t=\pi$ ⇐ ONLY SOLUTION IN $0 \leqslant t \leqslant 2\pi$

SETTING UP AN INTEGRAL

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt = \int_0^{\pi} (1+\cos t)(2t)\,dt$$

$$\text{AREA} = \int_0^{\pi} 2t(1+\cos t)\,dt$$

AS REQUIRED

b) INTEGRATION BY PARTS (IGNORING LIMITS)

$2t$	2
$t+\sin t$	$1+\cos t$

$$\int 2t(1+\cos t)\,dt$$

$$= 2t(t+\sin t) - 2\int t+\sin t\,dt$$

$$= 2t^2 - 2t\sin t - 2\left(\tfrac{1}{2}t^2 - \cos t\right) + C$$

$$= t^2 - 2t\sin t + 2\cos t + C$$

FINALLY THE AREA

$$\text{AREA} = \left[t^2 - 2t\sin t + 2\cos t\right]_0^{\pi} = (\pi^2 - 0 - 2) - (0 - 0 + 2)$$

$$= \pi^2 - 4$$

IYGB - MP2 PAPER I - QUESTION 4

SUPPOSE THAT THERE EXIST INTEGERS a & b SO THAT

$$a^2 - 8b = 7$$

THEN WE HAVE

$$a^2 = 8b + 7$$

AS THE R.H.S IS ODD (MULTIPLE OF 8 + 7), IMPLIES THAT a^2 IS ALSO ODD, AND THEREFORE a MUST ALSO BE ODD

LET $a = 2n+1$ WHICH IS ODD FOR n BEING AN INTEGER

$$\Rightarrow (2n+1)^2 = 8b + 7$$
$$\Rightarrow 4n^2 + 4n + 1 = 8b + 7$$
$$\Rightarrow 4n^2 + 4n - 8b = 6$$
$$\Rightarrow 2n^2 + 2n - 4b = 3$$
$$\Rightarrow 2(n^2 + n - 2b) = 3$$
$$\Rightarrow n^2 + n - 2b = \frac{3}{2}$$

BUT THE L.H.S HAS TO BE AN INTEGER, WHILE THE R.H.S IS NOT

$\therefore$ THIS IS A CONTRADICTION TO THE ASSERTION THAT THERE EXIST INTEGERS a & b WHICH SATISFIES $a^2 - 8b = 7$

IYGB - MP2 PAPER I - QUESTION 5

a) USING STANDARD METHODOLOGY

$$y = 4 - \frac{1}{x-1}$$

$$\frac{1}{x-1} = 4 - y$$

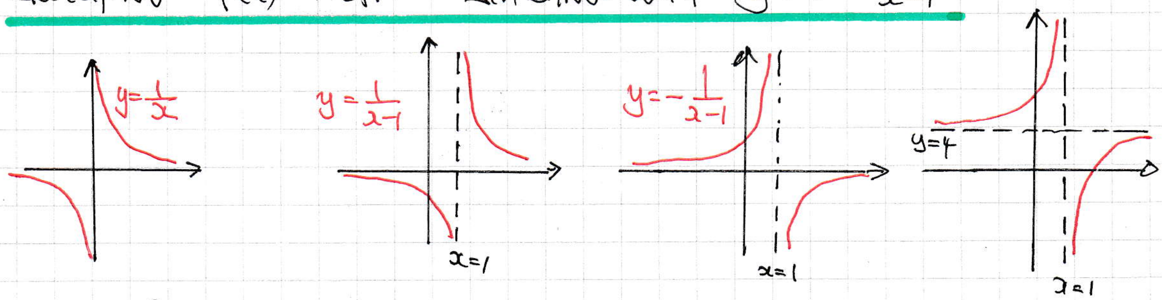
$$x - 1 = \frac{1}{4-y}$$

$$x = 1 + \frac{1}{4-y}$$

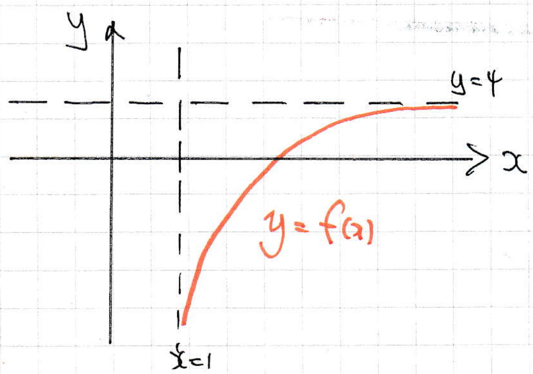
$$y = 1 + \frac{1}{4-x}$$

$$\therefore f^{-1}(x) = 1 - \frac{1}{x-4}$$

b) SKETCHING $f(x)$ FIRST - STARTING WITH $y = 4 - \frac{1}{x-1}$



NOW $f(x)$ WITH $x>1$



I.E RANGE OF $f(x)$ IS

$f(x) < 4$

	f	f^{-1}
D	$x>1$	$x<4$
R	$f(x)<4$	$f^{-1}(x)>1$

$\therefore$ DOMAIN $x<4$

RANGE $f^{-1}(x) > 1$

IYGB - MP2 PAPER I - QUESTION 6

a) FROM THE DEFINITION OF $g(x)$ & $f(x)$ AND USING $\sin(A \pm B)$

$$\Rightarrow g(x) = f\left(x+\frac{\pi}{4}\right) - f\left(x-\frac{\pi}{4}\right)$$

$$\Rightarrow g(x) = \sin\left[2\left(x+\frac{\pi}{4}\right)\right] - \sin\left[2\left(x-\frac{\pi}{4}\right)\right]$$

$$\Rightarrow g(x) = \sin\left[2x+\frac{\pi}{2}\right] - \sin\left[2x-\frac{\pi}{2}\right]$$

$$\Rightarrow g(x) = \cancel{\sin 2x \cos\frac{\pi}{2}} + \cos 2x \sin\frac{\pi}{2} - \left[\cancel{\sin 2x \cos\frac{\pi}{2}} - \cos 2x \sin\frac{\pi}{2}\right]$$

$$\Rightarrow g(x) = 2\cos 2x \sin\frac{\pi}{2} \qquad \sin\frac{\pi}{2} = 1$$

$$\Rightarrow g(x) = 2\cos 2x$$

AS REQUIRED

b) DIFFERENTIATING $g(x)$

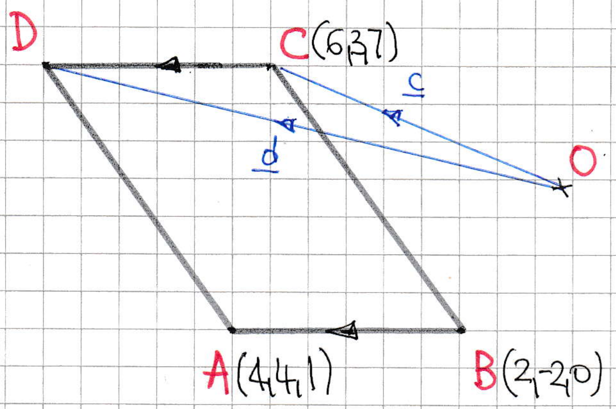
$$g(x) = 2\cos 2x$$

$$g'(x) = -4\sin 2x$$

$$g'(x) = -4f(x)$$

IYGB – MP2 PAPER I – QUESTION 7

LOOKING AT THE DIAGRAM BELOW



$\vec{OD} = \vec{OC} + \vec{CD}$

$\vec{OD} = \vec{OC} + \vec{BA}$

$\underline{d} = \underline{c} + (\underline{a} - \underline{b})$

$\underline{d} = \begin{pmatrix}6\\3\\7\end{pmatrix} + \begin{pmatrix}4\\4\\1\end{pmatrix} - \begin{pmatrix}2\\-2\\0\end{pmatrix}$

$\underline{d} = \begin{pmatrix}8\\9\\8\end{pmatrix}$

$D(8,9,8)$

NEXT CALCULATE SOME LENGTHS

$|\vec{OC}| = |\underline{c}| = |6,3,7| = \sqrt{36+9+49} = \sqrt{94}$

$|\vec{OD}| = |\underline{d}| = |8,9,8| = \sqrt{64+81+64} = \sqrt{209}$

$|\vec{CD}| = |\underline{d} - \underline{c}| = |(8,9,8) - (6,3,7)| = |2,6,1| = \sqrt{4+36+1} = \sqrt{41}$

BY THE COSINE RULE ON $O\hat{C}D$ – THE LARGEST ANGLE LIES OPPOSITE THE LONGEST LENGTH

$\Rightarrow |\vec{OD}|^2 = |\vec{CD}|^2 + |\vec{OC}|^2 - 2|\vec{CD}||\vec{OC}|\cos\theta$, WHERE $\theta = O\hat{C}D$

$\Rightarrow 209 = 41 + 94 - 2\sqrt{41}\sqrt{94}\cos\theta$

$\Rightarrow 2\sqrt{41}\sqrt{94}\cos\theta = -74$

$\Rightarrow \cos\theta = -\frac{37}{\sqrt{3854}}$

$\Rightarrow \theta \approx 126.6°$

IYGB - MP2 PAPER I - QUESTION 8

a) EXPANDING BINOMIALLY UP TO x^3

$$\Rightarrow \sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}(-x) + \frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}(-x)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\times 2\times 3}(-x)^3 + O(x^4)$$

$$\Rightarrow \sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + O(x^4)$$

b) PROCEED AS FOLLOWS

$$8 \times \sqrt{1 - \frac{1}{64}} = 8\sqrt{\frac{63}{64}} = \frac{8\sqrt{63}}{\sqrt{64}} = \frac{8\sqrt{9}\sqrt{7}}{\sqrt{64}}$$

$$= \frac{8 \times 3\sqrt{7}}{8} = 3\sqrt{7}$$

AS REQUIRED

c) COMBINING RESULTS

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x \quad \text{(FOR SMALL } x\text{, FIRST 2 TERMS)}$$

LET $x = \frac{1}{64}$

$$\Rightarrow \sqrt{1 - \frac{1}{64}} \approx 1 - \frac{1}{2} \times \frac{1}{64}$$

$$\Rightarrow 8\sqrt{1 - \frac{1}{64}} \approx 8\left[1 - \frac{1}{128}\right]$$

$$\Rightarrow 3\sqrt{7} \approx 8 - \frac{1}{16}$$

$$\Rightarrow 3\sqrt{7} \approx \frac{127}{16}$$

$$\Rightarrow \sqrt{7} \approx \frac{127}{48}$$

AS REQUIRED

IYGB – MP2 PAPER I – QUESTION 9

START BY OBTAINING THE GRADIENT FUNCTION

$$\Rightarrow ax^3 - 3xy + by^2 = 224$$

$$\Rightarrow \frac{d}{dx}\left(ax^3 - 3xy + by^2\right) = \frac{d}{dx}(224)$$

$$\Rightarrow 3ax^2 - 3y - 3x\frac{dy}{dx} + 2by\frac{dy}{dx} = 0$$

OBTAIN THE GRADIENT AT $(-2,6)$

$$\Rightarrow 3a(-2)^2 - 3\times 6 - 3(-2)\frac{dy}{dx} + 2b(6)\frac{dy}{dx} = 0$$

$$\Rightarrow 12a - 18 + 6\frac{dy}{dx} + 12b\frac{dy}{dx} = 0$$

$$\Rightarrow (6 + 12b)\frac{dy}{dx} = 18 - 12a$$

$$\Rightarrow \frac{dy}{dx} = \frac{18 - 12a}{6 + 12b}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3 - 2a}{1 + 2b}$$ ← GRADIENT OF THE TANGENT

$\therefore$ GRADIENT OF THE NORMAL IS $\frac{2b+1}{2a-3}$

NEXT WE REARRANGE THE NORMAL TO "READ" ITS GRADIENT

$$\Rightarrow 15x - 13y + 108 = 0$$

$$\Rightarrow 15x + 108 = 13y$$

$$\Rightarrow y = \frac{15}{13}x + \frac{108}{13}$$

↑

$$\frac{2b+1}{2a-3}$$

$$\Rightarrow \frac{2b+1}{2a-3} = \frac{15}{13}$$

IYGB - MP2 PAPER I - QUESTION 9

$$\Rightarrow 13(2b+1) = 15(2a-3)$$
$$\Rightarrow 26b + 13 = 30a - 45$$
$$\Rightarrow 26b - 30a = -58$$
$$\Rightarrow 30a - 26b = 58$$
$$\Rightarrow 15a - 13b = 29$$

THE POINT $(-2,6)$ MUST ALSO SATISFY THE EQUATION OF THE CURVE

$$\Rightarrow ax^3 - 3xy + by^2 = 224$$
$$\Rightarrow -8a - 3(-2)\times6 + b\times36 = 224$$
$$\Rightarrow -8a + 36 + 36b = 224$$
$$\Rightarrow -8a + 36b = 188$$
$$\Rightarrow -2a + 9b = 47$$

FINALLY SOLVING THE EQUATIONS SIMULTANEOUSLY

$$\left.\begin{array}{l} 15a - 13b = 29 \\ -2a + 9b = 47 \end{array}\right) \underset{\times 15}{\overset{\times 2}{\Rightarrow}} \left.\begin{array}{l} 30a - 26b = 58 \\ -30a + 135b = 705 \end{array}\right\} \Rightarrow$$

$$\Rightarrow 109b = 763$$
$$\Rightarrow b = 7$$

&
$$-2a + 9b = 47$$
$$-2a + 63 = 47$$
$$-2a = -16$$
$$a = 8$$

IYGB – MP2 PAPER I – QUESTION 10

a) LOOKING AT THE PATTERN

WEEK:	1	2	3	....	52
COMM.:	15	15 + 15 = 30	15 + 15 + 15 = 45		?

$a = 15$

$d = 15$

$n = 52$

$\Rightarrow U_n = a + (n-1)d$

$\Rightarrow U_{52} = 15 + 51 \times 15$

$\Rightarrow U_{52} = 780$ i.e. £780

b) SUMMING THE COMMISSIONS USING PART (a)

$S_n = \frac{n}{2}[a + L]$

$S_{52} = \frac{52}{2}[15 + 780] = 26 \times 795 = 20670$

i.e. £20670

c) CONTINUING THE PATTERN & LINKING WITH THE FIRST YEAR

WEEK	COMMISSION	
51	£765	FIRST YEAR
52	£780	
1	$(52 \times 10) + 20$	SECOND YEAR; $a = 540$
2	$(52 \times 10) + 20 + 20$	
3	$(52 \times 10) + 20 + 20 + 20$	
⋮	⋮	
52	$(52 \times 10) + 20 \times 52$	$L = 1560$

SUMMING USING $S_n = \frac{n}{2}[a + L]$

$S_{52} = \frac{52}{2}[540 + 1560] = 54600$ i.e. £54600

IYGB – MP2 PAPER I – QUESTION 11

USING THE SUBSTITUTION GIVEN

$$\int_0^{\frac{\pi}{4}} \frac{1-\tan^2 x}{\sec^2 x + 2\tan x}\,dx = \int_0^1 \frac{1-\tan^2 x}{\sec^2 x + 2\tan x}\left(\frac{du}{2\cos 2x}\right)$$

$u = \sin 2x$

$\frac{du}{dx} = 2\cos 2x$

$dx = \frac{du}{\cos 2x}$

$x = 0 \longmapsto u = 0$

$x = \frac{\pi}{4} \longmapsto u = 1$

$$= \frac{1}{2}\int_0^1 \frac{1-\tan^2 x}{(\sec^2 x + 2\tan x)\cos 2x}\,du$$

SWITCH EVERYTHING INTO SINES & COSINES

$$= \frac{1}{2}\int_0^1 \frac{1-\frac{\sin^2 x}{\cos^2 x}}{\left(\frac{1}{\cos^2 x}+\frac{2\sin x}{\cos x}\right)(\cos^2 x - \sin^2 x)}\,du$$

MULTIPLY "TOP & BOTTOM" OF THE DOUBLE FRACTION BY $\cos^2 x$

$$= \frac{1}{2}\int_0^1 \frac{\cancel{(\cos^2 x - \sin^2 x)}}{(1+2\sin x\cos x)\cancel{(\cos^2 x - \sin^2 x)}}\,du = \frac{1}{2}\int_0^1 \frac{1}{1+2\sin x\cos x}\,du$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+\sin 2x}\,du = \frac{1}{2}\int_0^1 \frac{1}{1+u}\,du$$

$$= \frac{1}{2}\Big[\ln|1+u|\Big]_0^1 = \frac{1}{2}\left[\ln 2 - \cancel{\ln 1}\right] = \frac{1}{2}\ln 2$$

IYGB - MP2 PAPER I - QUESTION 12

a) FORMING AN O.D.E.

$$\Rightarrow \frac{dV}{dt} = -kV$$

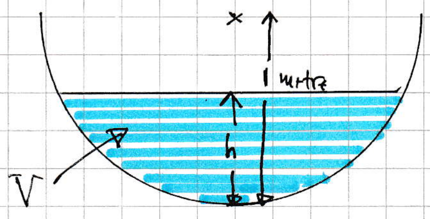
$$\Rightarrow \frac{dV}{dh} \times \frac{dh}{dt} = -kV$$

$$\Rightarrow \cancel{\pi}(2h-h^2)\frac{dh}{dt} = -k \times \frac{1}{3}\cancel{\pi}h^2(3-h)$$

$$\Rightarrow h(2-h)\frac{dh}{dt} = -\frac{1}{3}kh^2(3-h)$$

$$\Rightarrow (2-h)\frac{dh}{dt} = -\frac{1}{3}kh(3-h)$$

$$\Rightarrow \frac{dh}{dt} = -\frac{kh(3-h)}{3(2-h)}$$ // AS REQUIRED



$V = \frac{1}{3}\pi h^2(3-h)$

$V = \pi\left(h^2 - \frac{1}{3}h^3\right)$

$\frac{dV}{dh} = \pi\left(2h - h^2\right)$

b) SEPARATING VARIABLES

$$\Rightarrow \frac{3(2-h)}{h(3-h)}\,dh = -k\,dt$$

$$\Rightarrow \int \frac{3(2-h)}{h(3-h)}\,dh = \int -k\,dt$$

OBTAIN THE PARTIAL FRACTIONS

$$\frac{3(2-h)}{h(3-h)} \equiv \frac{A}{h} + \frac{B}{3-h}$$

$$3(2-h) \equiv A(3-h) + Bh$$

- IF $h=3$: $-3 = 3B$, $B = -1$
- IF $h=0$: $6 = 3A$, $A = 2$

$$\frac{3(2-h)}{h(3-h)} = \frac{2}{h} - \frac{1}{3-h}$$

IYGB - MP2 PAPER I - QUESTION 12

CARRYING OUT THE INTEGRATIONS

$$\Rightarrow \int \frac{2}{h} - \frac{1}{3-h} \, dh = -kt + C$$

$$\Rightarrow 2\ln|h| + \ln|3-h| = -kt + C$$

$$\Rightarrow \ln h^2 + \ln|3-h| = -kt + C$$

$$\Rightarrow \ln|h^2(3-h)| = -kt + C$$

$$\Rightarrow h^2(3-h) = e^{-kt+C}$$

$$\Rightarrow 3h^2 - h^3 = e^{-kt} \times e^{C}$$

$$\Rightarrow 3h^2 - h^3 = Ae^{-kt}$$

APPLY THE INITIAL CONDITION $t=0$, $h=1$ (INITIALLY FULL)

$$\Rightarrow 3 - 1 = Ae^0$$

$$\Rightarrow A = 2$$

$$\therefore \ 3h^2 - h^3 = 2e^{-kt}$$

AS REQUIRED