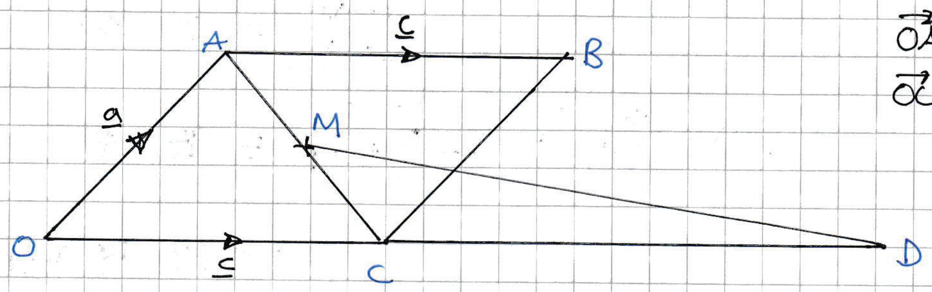
IYGB – SYNOPTIC PAPER K – QUESTION 1

a) START WITH A DIAGRAM



$\vec{OA} = \underline{a} = (7,4,3)$

$\vec{OC} = \underline{c} = (1,2,-1)$

$\vec{OB} = \vec{OA} + \vec{AB} = \vec{OA} + \vec{OC} = \underline{a} + \underline{c} = (7,4,3) + (1,2,-1) = (8,6,2)$

$B(8,6,2)$

b) PROCEED AS FOLLOWS

$\vec{AC} = \vec{AO} + \vec{OC} = -\underline{a} + \underline{c} = -(7,4,3) + (1,2,-1) = (-6,-2,-4)$

$\vec{MC} = \frac{1}{2}\vec{AC} = \frac{1}{2}(-6,-2,-4) = (-3,-1,-2)$

FINALLY WE HAVE

$\vec{MD} = \vec{MC} + \vec{CD}$

$(1,7,-6) = (-3,-1,-2) + \vec{CD}$

$\vec{CD} = (1,7,-6) - (-3,-1,-2)$

$\vec{CD} = (4,8,-4)$

$\vec{CD} = 4(1,2,-1)$

$\vec{CD} = 4(\vec{OC})$

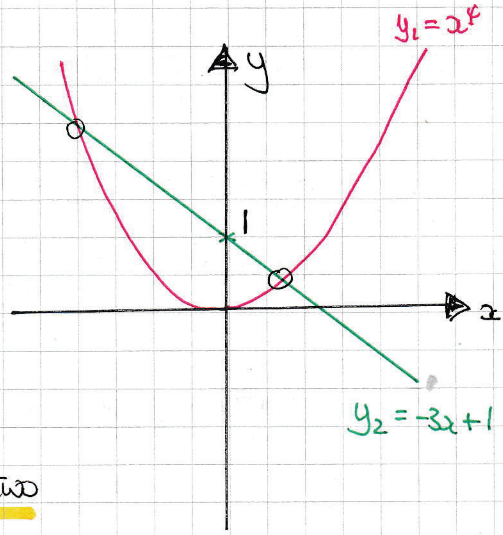
$\therefore$ RATIO $1:4$

IYGB – SYNOPTIC PAPER K – QUESTION 2

a) SPLIT INTO TWO GRAPHS

$$x^4 + 3x - 1 = 0$$
$$x^4 = -3x + 1$$

$$y_1 = x^4 \qquad y_2 = -3x + 1$$



∴ TWO REAL ROOTS AS THERE ARE TWO INTERSECTIONS

b) $f(x) = x^4 + 3x - 1$

$f(0) = -1 < 0$

$f(1) = 3 > 0$

AS $f(x)$ IS CONTINUOUS AND THERE IS A CHANGE OF SIGN IN THE INTERVAL $(0,1)$, THERE IS AT LEAST ONE ROOT IN THE INTERVAL

c) USING THE RECURRENCE RELATION GIVEN

$x_1 = 0.3306$
$x_2 = 0.3293$
$x_3 = 0.3294$
$x_4 = 0.3294$

(4 d.p)

d) THIS CAN BE REPRESENTED BY A COBWEB DIAGRAM, AS THE SUCCESSIVE APPROXIMATION OSCILLATE

IYGB - SYNOPTIC PAPER K - QUESTION K

e) $f(x) = x^4 + 3x - 1$

$f(0.329405) = -0.000011\ldots < 0$

$f(0.329415) = 0.000020\ldots > 0$

CONTINUITY & CHANGE OF SIGN IMPLY THE ROOT SATISFIES

$$0.329405 < \alpha < 0.329415$$

$$\therefore \quad \alpha = 0.32941$$

5 d.p

IYGB – SYNOPTIC PAPER K – QUESTION 3

a) I) THE POINT $C(10,0)$ IS SATISFIED BY BOTH THE LINE & CURVE

$y = q - 2x$
$0 = q - 2\times10$
$0 = q - 20$
$q = 20$

$y = p + 10x - x^2$
$0 = p + 10\times10 - 10^2$
$p = 0$

II) SOLVING SIMULTANEOUSLY

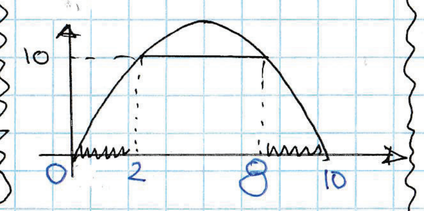
$$\left.\begin{array}{l} y = 20 - 2x \\ y = 10x - x^2 \end{array}\right\} \Rightarrow 20 - 2x = 10x - x^2$$
$\Rightarrow x^2 - 12x + 20 = 0$
$\Rightarrow (x-10)(x-2) = 0$
$\Rightarrow x = \begin{cases} 10 & \leftarrow C \\ 2 & \leftarrow A \end{cases}$
$\Rightarrow y = 10\times2 - 2^2 =$
$\therefore A(2,16)$

FINALLY USING THE QUADRATIC OR SYMMETRY

- $y = 16$

$16 = 10x - x^2$
$x^2 - 10x + 16 = 0$
$(x-2)(x-8) = 0$
$x = \begin{cases} \cancel{2} \\ 8 \end{cases}$

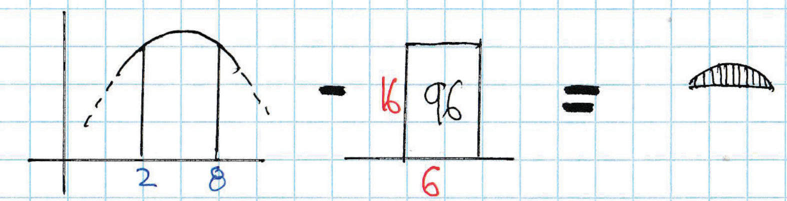
$\therefore (8,16)$



$y = 10x - x^2$
$y = x(10-x)$

$\therefore (8,16)$

b) LOOKING AT THE DIAGRAM BELOW



$$\int_2^8 10x - x^2\, dx = \left[5x^2 - \tfrac{1}{3}x^3\right]_2^8$$

IYGB - SYNOPTIC PAPER K - QUESTION 3

$$= \left(320 - \frac{512}{3}\right) - \left(20 - \frac{8}{3}\right)$$

$$= 132$$

$\therefore$ REQUIRED AREA $= 132 - 96 = 36$

IYGB - SYNOPTIC PAPER K - QUESTION 4

$$\boxed{k = 2^p - 1 \qquad N = k^2 - 1 \qquad \text{(GIVEN)}}$$

PROCEED BY DIRECT EVALUATION

$$N = k^2 - 1 = (2^p - 1)^2 - 1 = (2^p)^2 - 2 \times 2^p \times 1 + 1^2 - 1$$
$$= 2^{2p} - 2^{p+1} + 1 - 1$$
$$= 2^{2p} - 2^{p+1}$$
$$= 2^{p+1}(2^{p-1} - 1)$$

INDEED 2^{p+1} IS A FACTOR OF N

$$\left[\text{NOTE} \quad 2^{p+1} \times 2^{p-1} = 2^{p+1+p-1} = 2^{2p}\right]$$

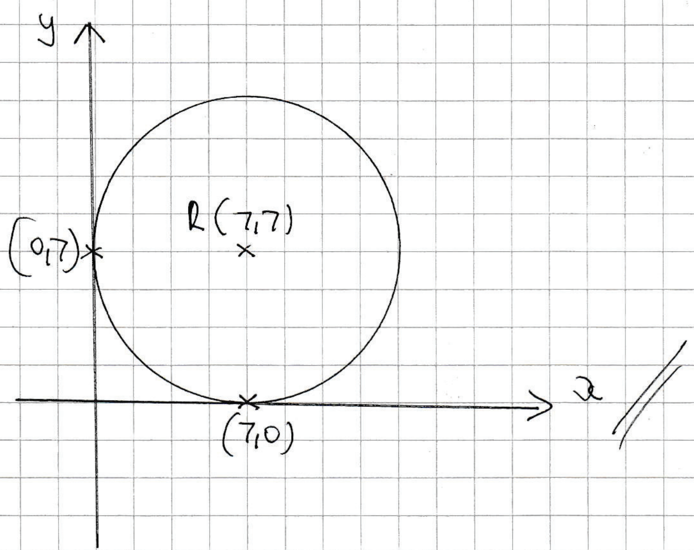
IYGB - SYNOPTIC PAPER K - QUESTION 5

a) COMPLETING THE SQUARE IN x & y

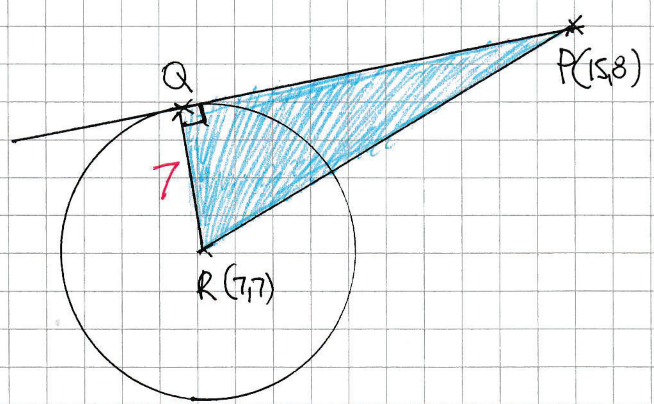
$$\Rightarrow x^2+y^2-14x-14y+49=0$$
$$\Rightarrow x^2-14x+y^2-14y+49=0$$
$$\Rightarrow (x-7)^2-49+(y-7)^2-\cancel{49}+\cancel{49}=0$$
$$\Rightarrow (x-7)^2+(y-7)^2=49$$

$\therefore$ CENTRE AT $R(7,7)$ & RADIUS IS 7

b) LOOKING AT THE INFORMATION ABOUT



c) LOOKING AT THE DIAGRAM BELOW – FIND THE LENGTH PR



- $|PR| = \sqrt{(8-7)^2+(15-7)^2}$

  $|PR| = \sqrt{1+64}$

  $|PR| = \sqrt{65}$

- BY PYTHAGORAS

  $|QR|^2+|QP|^2=|PR|^2$

  $7^2+|QP|^2=(\sqrt{65})^2$

  $49+|QP|^2=65$

  $|QP|^2=16$

  $|QP|=4$

IYGB - SYNOPTIC PAPER K - QUESTION 6

a) EXPAND BINOMIALLY UP TO a^3

$$(1+ax)^n = 1 + \frac{n}{1}(ax)^1 + \frac{n(n-1)}{1\times 2}(ax)^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}(ax)^3 + O(x^4)$$

$$(1+ax)^n = 1 + \underbrace{na}_{-10}x + \underbrace{\tfrac{1}{2}n(n-1)a^2}_{75}x^2 + \tfrac{1}{6}n(n-1)(n-2)a^3x^3 + O(x^4)$$

SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} na = -10 \\ \frac{1}{2}n(n-1)a^2 = 75 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} na = -10 \\ n(n-1)a^2 = 150 \end{array}\right\}$$

SQUARE

$\times n$

$$\Longrightarrow \left\{\begin{array}{l} n^2a^2 = 100 \\ (n-1)n^2a^2 = 150n \end{array}\right\}$$

$$\Longrightarrow 100(n-1) = 150n$$

$$\Longrightarrow 100n - 100 = 150n$$

$$\Longrightarrow -100 = 50n$$

$$\Longrightarrow n = -2$$

& SINCE $na = -10$

$a = 5$

b) SUBSTITUTING $a=5$, $n=-2$ INTO

$$\tfrac{1}{6}n(n-1)(n-2)a^3 = \tfrac{1}{6}(-2)(-3)(-4)\times 5^3 = -500$$

c) THE EXPANSION IS VALID FOR $|ax| < 1$

$$\Longrightarrow |5x| < 1$$

$$\Longrightarrow -\tfrac{1}{5} < x < \tfrac{1}{5}$$

IYGB - SYNOPTIC PAPER K - QUESTION 7

STARTING WITH THE FORMAL DEFINITION OF THE DERIVATIVE

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right] \quad \text{WITH } f(x) = \sin x$$

$$f'(x) = \lim_{h \to 0}\left[\frac{\sin(x+h) - \sin x}{h}\right]$$

USING COMPOUND ANGLE IDENTITIES

$$f'(x) = \lim_{h \to 0}\left[\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\right]$$

USING SMALL ANGLE APPROXIMATIONS

$$\sin h = h + O(h^3)$$
$$\cos h = 1 + O(h^2)$$

THUS WE OBTAIN:

$$f'(x) = \lim_{h \to 0}\left[\frac{\sin x[1 + O(h^2)] + \cos x[h + O(h^3)] - \sin x}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{\cancel{\sin x} + O(h^2)\sin x + h\cos x + O(h^3)\cos x - \cancel{\sin x}}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{O(h^2)\sin x + h\cos x + O(h^3)\cos x}{h}\right]$$

$$= \lim_{h \to 0}\left[O(h)\sin x + \cos x + O(h^2)\cos x\right]$$

$$= \cos x$$

AS REQUIRED

- 1 -

IYGB - SYNOPTIC PAPER K - QUESTION 8

START BY DIFFERENTIATING USING THE PRODUCT RULE

$$\Rightarrow y = x(\ln x)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = 1 \times (\ln x)^{\frac{1}{2}} + x \times \frac{1}{2}(\ln x)^{-\frac{1}{2}} \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = (\ln x)^{\frac{1}{2}} + \frac{1}{2}(\ln x)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}}$$

NOW WE REQUIRE $\frac{dy}{dx} = \frac{3}{2}$

$$\Rightarrow \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}} = \frac{3}{2}$$

$$\Rightarrow A + \frac{1}{2A} = \frac{3}{2} \qquad A = \sqrt{\ln x}$$

$$\Rightarrow 2A + \frac{1}{A} = 3$$

$$\Rightarrow 2A^2 + 1 = 3A$$

$$\Rightarrow 2A^2 - 3A + 1 = 0$$

$$\Rightarrow (2A - 1)(A - 1) = 0$$

$$\Rightarrow A = \begin{cases} 1/2 \\ 1 \end{cases}$$

$$\Rightarrow \sqrt{\ln x} = \begin{cases} 1/2 \\ 1 \end{cases}$$

$$\Rightarrow \ln x = \begin{cases} 1/4 \\ 1 \end{cases}$$

$$\Rightarrow x = \begin{cases} e^{1/4} \\ e \end{cases} \qquad y = \begin{cases} e^{\frac{1}{4}} \times \frac{1}{2} \\ e \times 1 \end{cases}$$

$$\therefore \left(e^{\frac{1}{4}}, \tfrac{1}{2}e^{\frac{1}{4}}\right) \text{ \& } (e, e)$$

IYGB – SYNOPTIC PAPER K – QUESTION 9

MODELLING AS FOLLOWS – LET THE "MIDDLE" TERM BE x

$$u_{n-2}, \quad u_{n-1}, \quad u_n, \quad u_{n+1}, \quad u_{n+2}$$

$$x-2d \quad x-d \quad x \quad x+d \quad x+2d$$

THE ARITHMETIC MEAN IS 7

$$\Rightarrow \frac{(x-2d)+(x-d)+x+(x+d)+(x+2d)}{5} = 7$$

$$\Rightarrow \frac{5x}{5} = 7$$

$$\Rightarrow x = 7$$

NEXT THE ARITHMETIC MEAN OF THE SQUARES IS 67

$$\Rightarrow \frac{(x-2d)^2+(x-d)^2+x^2+(x+d)^2+(x+2d)^2}{5} = 67$$

$$\Rightarrow (7-2d)^2+(7-d)^2+7^2+(7+d)^2+(7+2d)^2 = 67 \times 5$$

$$\Rightarrow 49-\cancel{28d}+4d^2+49-\cancel{14d}+d^2+49+49+\cancel{14d}+d^2+49+\cancel{28d}+4d^2 = 335$$

$$\Rightarrow 10d^2 + 49 \times 5 = 335$$

$$\Rightarrow 2d^2 + 49 = 67$$

$$\Rightarrow 2d^2 = 18$$

$$\Rightarrow d^2 = 9$$

$$\Rightarrow d = \pm 3$$

$\therefore$ IF $d=3$ THE NUMBERS ARE 1, 4, 7, 10, 13

(IF $d=-3$ THE NUMBERS ARE 13, 10, 7, 4, 1)

IYGB - SYNOPTIC PAPER K - QUESTION 10

PROCEED AS FOLLOWS

$$\frac{2^n}{2^{\sqrt{n}} \times 2^6} = 1 \implies \frac{2^n}{2^{\sqrt{n}+6}} = 1$$

$$\implies 2^n = 2^{\sqrt{n}+6}$$

$$\implies n = \sqrt{n} + 6$$

THIS IS A QUADRATIC IN $\sqrt{n}$

$$\implies (\sqrt{n})^2 - \sqrt{n} - 6 = 0$$

$$\implies (\sqrt{n} - 3)(\sqrt{n} + 2) = 0$$

$$\implies \sqrt{n} = \begin{cases} 3 \\ \cancel{-2} \end{cases}$$

$$\implies n = 9$$

IYGB - SYNOPTIC PAPER K - QUESTION 11

METHOD A - BY COMPLETING THE SQUARE

$\Rightarrow 5x^2 - 9x - 1 = 0$

$\Rightarrow x^2 - \frac{9}{5}x - \frac{1}{5} = 0$

$\Rightarrow x^2 - 1.8x - 0.2 = 0$

$\Rightarrow (x-0.9)^2 - (0.9)^2 - 0.2 = 0$

$\Rightarrow (x-0.9)^2 - 0.81 - 0.2 = 0$

$\Rightarrow (x-0.9)^2 = 1.01$

$\Rightarrow x - 0.9 = \pm\sqrt{1.01}$

$\Rightarrow x - 0.9 \simeq \begin{cases} 1 \\ -1 \end{cases}$

$\Rightarrow x \approx \begin{cases} 1.9 \\ -0.1 \end{cases}$ to 1 d.p

METHOD B - BY THE QUADRATIC FORMULA

$5x^2 - 9x - 1 = 0$

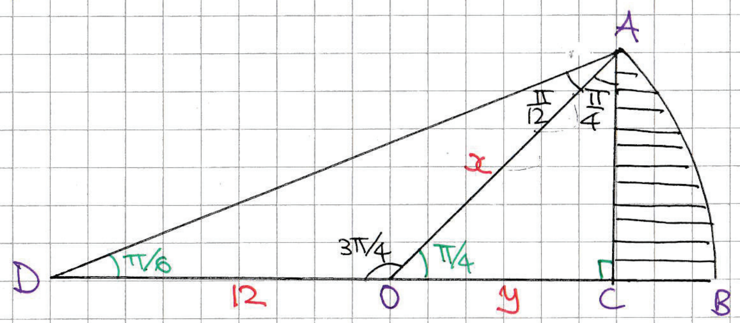
$$x = \frac{+9 \pm \sqrt{(-9)^2 - 4\times5\times(-1)}}{2\times5}$$

$$x = \frac{9 \pm \sqrt{81+20}}{10}$$

$$x = \frac{9 \pm \sqrt{101}}{10} = \begin{cases} \frac{9+\sqrt{101}}{10} \approx \frac{9+10}{10} = \frac{19}{10} = 1.9 \\ \frac{9-\sqrt{101}}{10} \approx \frac{9-10}{10} = \frac{-1}{10} = -0.1 \end{cases}$$

IYGB - SYNOPTIC PAPER K - QUESTION 12

a) STARTING WITH A DIAGRAM & OBTAIN SOME ANGLES



- $D\hat{O}A = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ (straight line)
- $D\hat{A}O = \pi - \left(\frac{\pi}{6} + \frac{3\pi}{4}\right) = \frac{\pi}{12}$ (triangle $D\hat{A}O$)
- $O\hat{A}C = \pi - \left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \frac{\pi}{4}$ (triangle $O\hat{A}C$)

BY THE SINE RULE ON $\hat{AOD}$

$$\frac{|OA|}{\sin\pi/6} = \frac{|OD|}{\sin\pi/12} \Longrightarrow \frac{x}{\sin\frac{\pi}{6}} = \frac{12}{\sin\frac{\pi}{12}}$$

$$\Longrightarrow x = \frac{12\sin\pi/6}{\sin\pi/12}$$

$$\Longrightarrow x = 23.18221983\ldots$$

AREA OF SECTOR $=$ "$\frac{1}{2}r^2\theta$" $= \frac{1}{2}x^2 \times \frac{\pi}{4}$

$$= \frac{1}{2}(23.1822\ldots)^2 \times \frac{\pi}{4}$$

$$= 211.0$$

b) NEXT LOOKING AT $\hat{AOC}$

$$\frac{|OC|}{|OA|} = \cos\frac{\pi}{4} \Longrightarrow \frac{y}{x} = \cos\frac{\pi}{4}$$

$$\Longrightarrow y = x\cos\frac{\pi}{4}$$

$$\Longrightarrow y = (23.1822\ldots) \times \frac{\sqrt{2}}{2}$$

$$\Longrightarrow y \approx 16.39230480\ldots$$

THE AREA OF THE TRIANGLE $\hat{AOC}$ IS GIVEN BY

$$\text{AREA} = \frac{1}{2}|OA||OC|\sin\frac{\pi}{4}.$$

$$= \frac{1}{2}xy \times \frac{\sqrt{2}}{2} = \frac{1}{2}(23.1822\ldots)(16.3923\ldots) \times \frac{\sqrt{2}}{2}$$

$$= 134.3538291\ldots$$

FINALLY THE SHADED AREA CAN BE FOUND

AREA OF SECTOR − AREA OF TRIANGLE

$= 211.0 - 134.3538\ldots$

$= 76.69\ldots$

$\simeq 77$

AS REQUIRED

IYGB – SYNOPTIC PAPER K – QUESTION 13

a) IF $t=1$

$P(1,1)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{-\frac{1}{t^2}} = -2t^3$$

$$\left.\frac{dy}{dx}\right|_{t=1} = -2$$

EQUATION OF TANGENT IS

$$y - y_0 = m(x - x_0)$$
$$y - 1 = -2(x-1)$$
$$y - 1 = -2x + 2$$
$$y + 2x = 3$$

AS REQUIRED

b) SOLVING SIMULTANEOUSLY

$x = \frac{1}{t}$, $y = t^2$ AND $y + 2x = 3$

$$\Rightarrow t^2 + 2\left(\frac{1}{t}\right) = 3$$
$$\Rightarrow t^2 + \frac{2}{t} = 3$$
$$\Rightarrow t^3 + 2 = 3t$$
$$\Rightarrow t^3 - 3t + 2 = 0$$
$$\Rightarrow (t-1)^2(t+2) = 0$$
$$\Rightarrow t = \begin{cases} 1 \leftarrow P \\ -2 \leftarrow Q \end{cases}$$

$$\therefore Q\left(-\frac{1}{2}, 4\right)$$

$t=1$ (REPEATED) MUST BE A SOLUTION FROM THE POINT OF TANGENCY

$$\left.\begin{aligned} &(t+2)(t-1)^2 \\ =\ &(t+2)(t^2-2t+1) \\ =\ &t^3 - \cancel{2t^2} + t \\ &\quad + \cancel{2t^2} - 4t + 2 \\ &t^3 - 3t + 2 \end{aligned}\right\} \text{CHECK}$$

IYGB – SYNOPTIC PAPER K – QUESTION 14

a) DIFFERENTIATE WITH RESPECT TO x

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(4x) + \frac{d}{dx}(4y) - \frac{d}{dx}(20)$$

$$\Rightarrow 2x + 2y + 2x\frac{dy}{dx} - 6y\frac{dy}{dx} = 4 + 4\frac{dy}{dx} - 0$$

$$\Rightarrow 2x + 2y - 4 = 6y\frac{dy}{dx} - 2x\frac{dy}{dx} + 4\frac{dy}{dx}$$

$$\Rightarrow 2x + 2y - 4 = (6y - 2x + 4)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + 2y - 4}{6y - 2x + 4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x + y - 2}{3y - x + 2}$$ // AS REQUIRED

b) SOLVING $\frac{dy}{dx} = 0$

$$\frac{x + y - 2}{3y - x + 2} = 0$$

$$x + y - 2 = 0$$

$$y = 2 - x$$

SUBSTITUTE INTO THE EQUATION OF THE CURVE

$$\Rightarrow x^2 + 2x(2-x) - 3(2-x)^2 = 4x + 4(2-x) - 20$$

$$\Rightarrow x^2 + \cancel{4x} - 2x^2 - \cancel{12} + 12x - 3x^2 = \cancel{4x} + \cancel{8} - 4x - \cancel{20}$$

$$\Rightarrow 0 = 4x^2 - 16x$$

$$\Rightarrow 0 = x^2 - 4x$$

$$\Rightarrow 0 = x(x-4)$$

$$x = \begin{cases} 0 \\ 4 \end{cases} \qquad y = \begin{cases} 2 \\ -2 \end{cases} \qquad \therefore (0,2) \text{ \& } (4,-2)$$ //

IYGB - SYNOPTIC PAPER K - QUESTION 14

c) STARTING WITH

$$2x+2y-4=(6y-2x+4)\frac{dy}{dx}$$
$$x+y-2=(3y-x+2)\frac{dy}{dx}$$
$\div 2$

$\longleftarrow$ PRODUCT RULE

DIFFERENTIATE AGAIN W.R.T x

$$1+\frac{dy}{dx}-0=\left(3\frac{dy}{dx}-1+0\right)\frac{dy}{dx}+(3y-x+2)\frac{d^2y}{dx^2}$$

$$1+\frac{dy}{dx}=3\left(\frac{dy}{dx}\right)^2-\frac{dy}{dx}+(3y-x+2)\frac{d^2y}{dx^2}$$

$$(x-3y-2)\frac{d^2y}{dx^2}-3\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}+1=0$$

AS REQUIRED

d) CHECKING $(0,2)$ & NOTE $\frac{dy}{dx}=0$ AT $(0,2)$

$$(0-6-2)\frac{d^2y}{dx^2}+1=0$$

$$\frac{d^2y}{dx^2}=\frac{1}{8}>0$$

$\therefore$ $(0,2)$ IS A LOCAL MIN

CHECKING $(4,-2)$ & $\frac{dy}{dx}=0$ AT $(4,-2)$

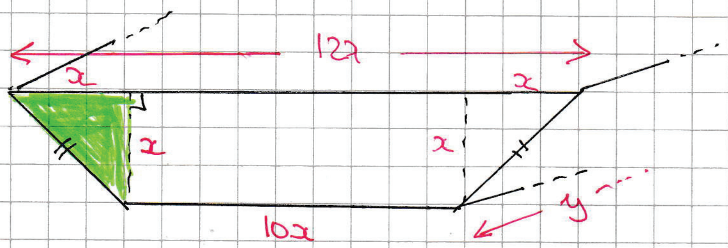
$$(4+6-2)\frac{d^2y}{dx^2}+1=0$$

$$\frac{d^2y}{dx^2}=-\frac{1}{8}<0$$

$\therefore$ $(4,-2)$ IS A LOCAL MAX

IYGB - SYNOPTIC PAPER K - QUESTION 15

a)



$d^2 = x^2 + x^2$

$d^2 = 2x^2$

$d = \sqrt{2}x$

THE TOTAL SURFACE AREA

$$\Rightarrow A = \underset{\text{FLOOR}}{10xy} + 2\underset{\text{TRAPEZOIDAL SIDES}}{\left[\frac{12x+10x}{2}x\right]} + \underset{\text{SLOPING SIDES}}{2dy}$$

$$\Rightarrow A = 10xy + 22x^2 + 2y(\sqrt{2}x)$$

$$\Rightarrow A = 22x^2 + 10xy + 2\sqrt{2}xy$$

$$\Rightarrow A = 22x^2 + (10 + 2\sqrt{2})xy$$

$$\Rightarrow A = 22x^2 + (10 + 2\sqrt{2})\left(\frac{180}{x}\right)$$

$$\Rightarrow A = 22x^2 + \frac{360}{x}(5 + \sqrt{2})$$

AS REQUIRED

CONSTRAINT ON CAPACITY

$V = 1980$

$$1980 = \underbrace{\left(\frac{12x+10x}{2}\right)x}_{\text{AREA OF TRAPEZIUM ABFE}} \times y$$

$11x^2y = 1980$

$x^2y = 180$

$x(xy) = 180$

$xy = \frac{180}{x}$

b) DIFFERENTIATE THE ABOVE EQUATION & SOLVING FOR ZERO

$$\Rightarrow A = 22x^2 + 360(5+\sqrt{2})x^{-1}$$

$$\Rightarrow \frac{dA}{dx} = 44x - 360(5+\sqrt{2})x^{-2}$$

IYGB - SYNOPTIC PAPER K - QUESTION 15

$$\Rightarrow \frac{dA}{dx} = 44x - \frac{360(5+\sqrt{2})}{x^2}$$

SOLVING FOR ZERO

$$\Rightarrow 0 = 44x - \frac{360(5+\sqrt{2})}{x^2}$$

$$\Rightarrow 0 = 44x^3 - 360(5+\sqrt{2})$$

$$\Rightarrow x^3 = \frac{90(5+\sqrt{2})}{11}$$

$$\Rightarrow x \approx 3.744$$

USING THE SECOND DERIVATIVE

$$\Rightarrow \frac{d^2A}{dx^2} = 44 + 720(5+\sqrt{2})x^{-3}$$

$$\Rightarrow \frac{d^2A}{dx^2} = 132 > 0$$

INDEED $x \approx 3.744$ MINIMIZES A

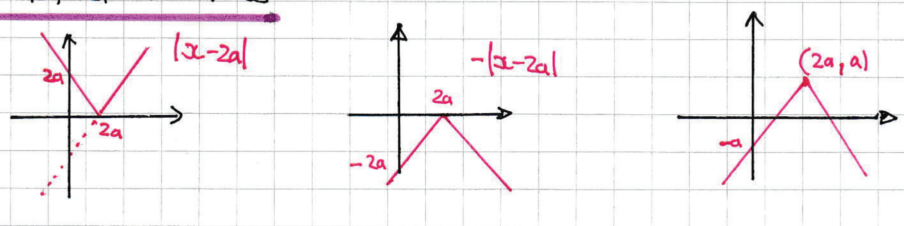
c) $A = 22x^2 + \frac{360}{x}(5+\sqrt{2})$

$$\Rightarrow A_{MIN} = 22(3.744)^2 + \frac{360}{3.744}(5+\sqrt{2})$$

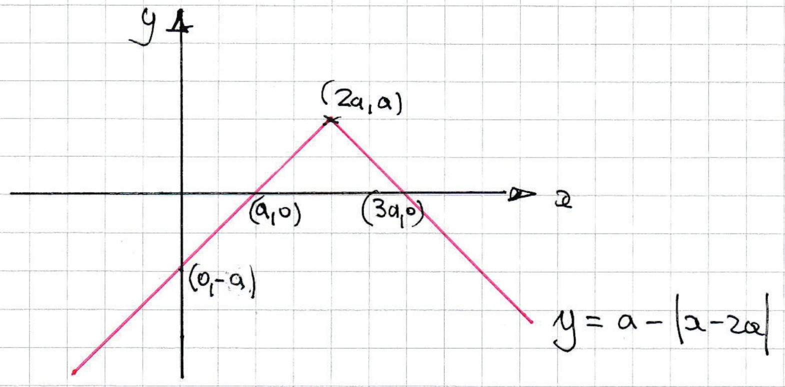
$$\Rightarrow A_{MIN} \simeq 925 \text{ cm}^3$$

IYGB - SYNOPTIC PAPER K - QUESTION 16

a) BY TRANSFORMATIONS

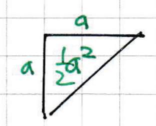


HENCE WE CAN SKETCH (x CO-ORDINATES BY INSPECTION)



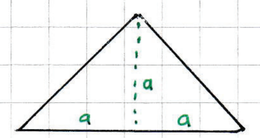
b) LOOKING AT THE GRAPH ABOVE

$$\int_0^{3a} f(x)\,dx =$$



UNDER THE x AXIS

$+$



ABOVE x AXIS

$$= -\frac{1}{2}a^2 + a^2$$

$$= \frac{1}{2}a^2$$

IYGB - SYNOPTIC PAPER K - QUESTION 17

a) $f(x) = x^2 - 4x + 1 \quad x > 4$

$y = x^2 - 4x + 1$

$y = (x-2)^2 - 3$

$y + 3 = (x-2)^2$

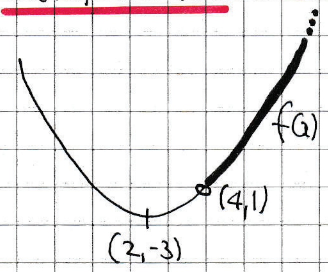
$x - 2 = \pm\sqrt{y+3}$

$x - 2 = +\sqrt{y+3} \quad (x > 4)$

$x = 2 + \sqrt{y+3}$

$f^{-1}(x) = 2 + \sqrt{x+3}$ //

b) SKETCHING $f(x)$



	f	f^{-1}
D	$x > 4$	$x > 1$
R	$f(x) > 1$	$f^{-1}(x) > 4$

$\therefore$ DOMAIN OF $f^{-1}(x)$: $x > 1$

RANGE OF $f^{-1}(x)$: $f^{-1}(x) > 4$

c) USING THE FACT THAT $f^{-1}(x) = f(x)$ CAN BE SOLVED AS $f(x) = x$ OR $f^{-1}(x) = x$ (IF IT IS EASIER) WE HAVE:

$\Rightarrow x^2 - 4x + 1 = x \quad \leftarrow \boxed{f(x) = x}$

$\Rightarrow x^2 - 5x + 1 = 0$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 1 = 0$

$\Rightarrow \left(x - \frac{5}{2}\right)^2 = \frac{21}{4}$

$\Rightarrow \left(x - \frac{5}{2}\right) = \pm\frac{\sqrt{21}}{2}$

$\Rightarrow x = \frac{5}{2} \pm \frac{\sqrt{21}}{2}$

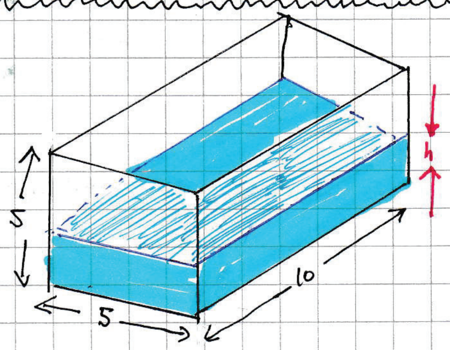
BUT AS $x > 4$

$\Rightarrow x = \frac{5 + \sqrt{21}}{2}$ //

IYGB - SYNOPTIC PAPER K - QUESTION 18

a) **FORMING AN O.D.E**

- IN: $\frac{dV}{dt} = 50$
- OUT $\frac{dV}{dt} = -10h$
- NET $\frac{dV}{dt} = 50 - 10h$



V = VOLUME OF WATER

$V = 5 \times 10 \times h$

$V = 50h$

$\frac{dV}{dh} = 50$

RELATING V & h

$$\frac{dV}{dh} \times \frac{dh}{dt} = 50 - 10h$$

$$50 \times \frac{dh}{dt} = 50 - 10h$$

$$5\frac{dh}{dt} = 5 - h$$

// AS REQUIRED

b) **SOLVING BY SEPARATION OF VARIABLES**

$$\Rightarrow 5\,dh = (5-h)\,dt$$

$$\Rightarrow \int \frac{1}{5-h}\,dh = \int \frac{1}{5}\,dt$$

$$\Rightarrow -\ln|5-h| = \frac{1}{5}t + C$$

$$\Rightarrow \ln|5-h| = -\frac{1}{5}t + C$$

$$\Rightarrow 5-h = e^{-\frac{1}{5}t+C}$$

$$\Rightarrow 5-h = e^{-\frac{1}{5}t} \times e^{C}$$

$$\Rightarrow 5-h = Ae^{-\frac{1}{5}t} \qquad (A = e^{C})$$

$$\Rightarrow 5 + Ae^{-\frac{1}{5}t} = h$$

IYGB - SYNOPTIC PAPER K - QUESTION 18

$$\Longrightarrow \boxed{h = 5 + Ae^{-\frac{1}{5}t}}$$

WHEN $t=0$ $h=2$ (ARBITRARY START OF CLOCK)

$$2 = 5 + Ae^{0}$$
$$2 = 5 + A$$
$$A = -3$$

$$\therefore \boxed{h = 5 - 3e^{-\frac{1}{5}t}}$$

WHEN $h=4$

$$4 = 5 - 3e^{-\frac{1}{5}t}$$
$$3e^{-\frac{1}{5}t} = 1$$
$$e^{-\frac{1}{5}t} = \frac{1}{3}$$
$$e^{\frac{1}{5}t} = 3$$
$$\frac{1}{5}t = \ln 3$$
$$t = 5\ln 3$$

IYGB - SYNOPTIC PAPER K - QUESTION 19

LET $\theta = \arctan\left(\frac{3}{x}\right)$ & $\phi = \arctan\left(\frac{2x}{9}\right)$

$$\Rightarrow 2\arctan\left(\frac{3}{x}\right) = \arctan\left(\frac{2x}{9}\right)$$

$$\Rightarrow 2\theta = \phi$$

$$\Rightarrow \tan 2\theta = \tan\phi$$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \tan\phi$$

BUT IF $\theta = \arctan\left(\frac{3}{x}\right) \Rightarrow \tan\theta = \frac{3}{x}$

$\phi = \arctan\left(\frac{2x}{9}\right) \Rightarrow \tan\phi = \frac{2x}{9}$

$$\Rightarrow \frac{2\left(\frac{3}{x}\right)}{1-\left(\frac{3}{x}\right)^2} = \frac{2x}{9}$$

$$\Rightarrow \frac{\frac{6}{x}}{1-\frac{9}{x^2}} = \frac{2x}{9}$$

MULTIPLY "TOP" & "BOTTOM" OF THE DOUBLE FRACTION BY x^2

$$\Rightarrow \frac{6x}{x^2-9} = \frac{2x}{9}$$

AS $x \neq 0$, WE MAY DIVIDE BOTH SIDES BY x

$$\Rightarrow \frac{6}{x^2-9} = \frac{2}{9}$$

$$\Rightarrow 2(x^2-9) = 54$$

$$\Rightarrow x^2 - 9 = 27$$

$$\Rightarrow x^2 = 36$$

$$\Rightarrow x = \pm 6$$

IYGB – SYNOPTIC PAPER K – QUESTION 20

TIDY USING RULES OF INDICES

$$\Rightarrow 4^{x+1} \times 3^{1-2x} = 24$$

$$\Rightarrow 4 \times 4^{x} \times 3 \times 3^{-2x} = 24$$

$$\Rightarrow 12 \times 4^{x} \times 3^{-2x} = 24$$

$$\Rightarrow 4^{x} \times 3^{-2x} = 2$$

$$\Rightarrow 4^{x} \times (3^{2})^{-x} = 2$$

$$\Rightarrow 4^{x} \times 9^{-x} = 2$$

$$\Rightarrow \left(\frac{4}{9}\right)^{x} = 2$$

$$\Rightarrow \log\left(\frac{4}{9}\right)^{x} = \log 2$$

$$\Rightarrow x\log\left(\frac{4}{9}\right) = \log 2$$

$$\Rightarrow x = \frac{\log 2}{\log 4/9} \approx -0.854$$

ALTERNATIVE, BY TAKING LOGS STRAIGHT AWAY

$$\Rightarrow 4^{x+1} \times 3^{1-2x} = 24$$

$$\Rightarrow \log\left(4^{x+1} \times 3^{1-2x}\right) = \log 24$$

$$\Rightarrow \log 4^{x+1} + \log 3^{1-2x} = \log 24$$

$$\Rightarrow (x+1)\log 4 + (1-2x)\log 3 = \log 24$$

$$\Rightarrow x\log 4 + \log 4 + \log 3 - 2x\log 3 = \log 24$$

$$\Rightarrow x\log 4 - 2x\log 3 = \log 24 - \log 3 - \log 4$$

$$\Rightarrow x\left[\log 4 - 2\log 3\right] = \log\left(\frac{24}{3\times 4}\right)$$

IYGB - SYNOPTIC PAPER K - QUESTION 20

$$\Longrightarrow x(\log 4 - \log 9) = \log 2$$

$$\Longrightarrow x = \frac{\log 2}{\log 4 - \log 9}$$

$$\Longrightarrow x \simeq -0.854$$

AS BEFORE

IYGB - SYNOPTIC PAPER K - QUESTION 21

A: $\sqrt{x} + \sqrt{y} = 1$ (GIVEN)

- B IS A REFLECTION OF A ABOUT THE y AXIS

$\Rightarrow$ REPLACE x FOR $-x$.

$\Rightarrow \sqrt{-x} + \sqrt{y} = 1$

- C IS A REFLECTION OF B ABOUT THE x AXIS

$\Rightarrow$ REPLACE y FOR $-y$ IN THE ABOVE EQUATION

$\Rightarrow \sqrt{-x} + \sqrt{-y} = 1$

- D IS A REFLECTION OF A IN THE x AXIS

$\Rightarrow$ REPLACE y FOR $-y$ IN THE "A" EQUATION

$\Rightarrow \sqrt{x} + \sqrt{-y} = 1$

IYGB - SYNOPTIC PAPER K - QUESTION 22

PROCEED BY SPLITTING THE FRACTION

$$\int_{\pi/8}^{\pi/4} \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\, dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^6 x}{\sin^2 x \cos^2 x} + \frac{\cos^6 x}{\sin^2 x \cos^2 x}\, dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos^2 x} + \frac{\cos^4 x}{\sin^2 x}\, dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{(1-\cos^2 x)^2}{\cos^2 x} + \frac{(1-\sin^2 x)^2}{\sin^2 x}\, dx$$

EXPANDING & SPLIT THE FRACTIONS AGAIN

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1-2\cos^2 x+\cos^4 x}{\cos^2 x} + \frac{1-2\sin^2 x+\sin^4 x}{\sin^2 x}\, dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x - 2 + \cos^2 x + \operatorname{cosec}^2 x - 2 + \sin^2 x\, dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x + \operatorname{cosec}^2 x + (\cos^2 x + \sin^2 x) - 4\, dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sec^2 x + \operatorname{cosec}^2 x - 3\, dx$$

$$= \Big[\tan x - \cot x - 3x \Big]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

IYGB - SYNOPTIC PAPER K - QUESTION 22

TIDY BEFORE EVALUATING

$$= \left[\tan x - \frac{1}{\tan x} - 3x \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= \left[\frac{\tan^2 x - 1}{\tan x} - 3x \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= -\left[3x + \frac{1 - \tan^2 x}{\tan x} \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= \left[3x + 2\left(\frac{1 - \tan^2 x}{2\tan x} \right) \right]_{\frac{\pi}{4}}^{\frac{\pi}{8}}$$

$$= \left[3x + 2\cot 2x \right]_{\frac{\pi}{4}}^{\frac{\pi}{8}}$$

$$= \left[3x + \frac{2}{\tan 2x} \right]_{\frac{\pi}{4}}^{\frac{\pi}{8}}$$

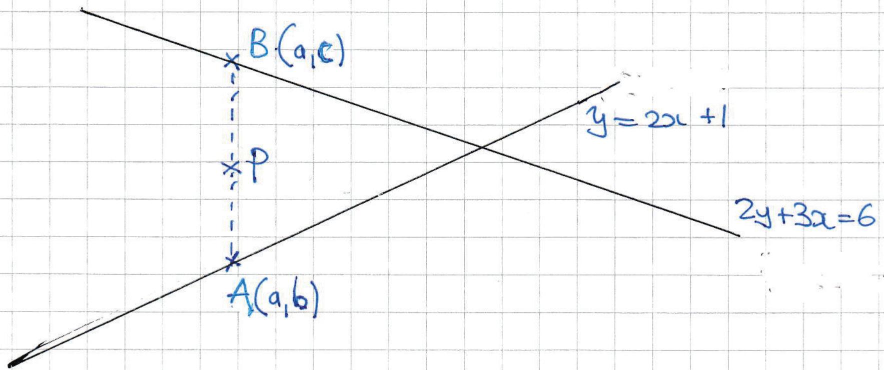
$$= \left(\frac{3\pi}{8} + \frac{2}{\tan\frac{\pi}{4}} \right) - \left(\frac{3\pi}{4} + \cancel{\frac{2}{\tan\frac{\pi}{2}}} \right)$$

$$= -\frac{3\pi}{8} + 2$$

$$= \frac{1}{8}(16 - 3\pi)$$

IYGB - SYNOPTIC PAPER K - QUESTION 23

STARTING WITH A DIAGRAM



EVIDENTLY P IS THE MIDPOINT OF AB

- $A(a,b)$ LIES ON $y = 2x+1$

$\Rightarrow \underline{b = 2a+1}$

- $B(a,c)$ LIES ON $2y+3x=6$

$\Rightarrow 2c + 3a = 6$

$\Rightarrow c + \frac{3}{2}a = 3$

$\Rightarrow \underline{c = 3 - \frac{3}{2}a}$

THUS P HAS y CO-ORDINATE

$$\text{"}y\text{"} = \frac{1}{2}(b+c) = \frac{1}{2}\left[2a+1+3-\frac{3}{2}a\right] = \frac{1}{2}\left(\frac{1}{2}a+4\right) = \frac{1}{4}a+2$$

THUS THE POINT P HAS GENERAL CO-ORDINATES $(a, \frac{1}{4}a+2)$

$$\Rightarrow \begin{cases} x = a \\ y = \frac{1}{4}a+2 \end{cases}$$

$$\Rightarrow y = \frac{1}{4}x+2$$