IYGB - SYNOPTIC PAPER V - QUESTION 1

FORM 3 LINEAR EQUATIONS FROM THE INFORMATION GIVEN

- $f(1)=1$
  $a+b+c=1$
- $f(2)=2$
  $4a+2b+c=2$
- $f(-2)=70$
  $4a-2b+c=70$

ELIMINATE $c=1-a-b$ FROM THE FIRST EQUATION & SUBSTITUTE INTO THE OTHER TWO

$$\left.\begin{array}{l}4a+2b+(1-a-b)=2\\4a-2b+(1-a-b)=70\end{array}\right\} \Rightarrow \begin{array}{l}3a+b+1=2\\3a-3b+1=70\end{array}$$

$$\Rightarrow \begin{array}{l}3a+b=1\\3a-3b=69\end{array}$$

SUBTRACTING GIVES

$$4b=-68$$
$$b=-17$$

$$3a+b=1$$
$$3a-17=1$$
$$3a=18$$
$$a=6$$

$$c=1-a-b$$
$$c=1-6-(-17)$$
$$c=-5+17$$
$$c=12$$

IYGB - SYNOPTIC PAPER V - QUESTION 2

a) $16^{\frac{1}{2}} + 16^{-\frac{3}{4}} = \sqrt{16} + \frac{1}{(\sqrt[4]{16})^3} = 4 + \frac{1}{8} = \frac{33}{8}$

$16^{-\frac{3}{4}} \to \left(\frac{1}{16^{3/4}}\right)$

b) $x^{-\frac{2}{3}} = 64$

$\left(x^{-\frac{2}{3}}\right)^{-\frac{3}{2}} = 64^{-\frac{3}{2}}$

$x^{1} = \frac{1}{64^{3/2}}$

$x = \frac{1}{(\sqrt{64})^3} = \frac{1}{8^3} = \frac{1}{512}$

c) $\left(x^{\frac{3}{2}} + 2x^{-\frac{3}{2}}\right)^2 = \left(x^{\frac{3}{2}} + 2x^{-\frac{3}{2}}\right)\left(x^{\frac{3}{2}} + 2x^{-\frac{3}{2}}\right)$

$= x^{\frac{3}{2}}x^{\frac{3}{2}} + 2x^{-\frac{3}{2}}x^{\frac{3}{2}} + 2x^{-\frac{3}{2}}x^{\frac{3}{2}} + 4x^{-\frac{3}{2}}x^{-\frac{3}{2}}$

$= x^3 + 2x^0 + 2x^0 + x^{-3}$

$= x^3 + 4 + \frac{4}{x^3}$

IYGB - SYNOPTIC PAPER V - QUESTION 3

a) USING PASCALS TRIANGLE OR A CALCULATOR

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$(5+10x)^4 = 1\times5^4\times(10x)^0 + 4\times5^3\times(10x)^1 + 6\times5^2\times(10x)^2 + 4\times5^1\times(10x)^3 + 1\times5^0\times(10x)^4$$

$$= 625 + 4\times125\times10x + 6\times25\times100x^2 + 20\times1000x^3 + 10000x^4$$

$$(5+10x)^4 = 625 + 5000x + 15000x^2 + 20000x^3 + 10000x^4$$

b) LET $x=100$ IN THE ABOVE EXPANSION

$$(5+10\times100)^4 = 625 + 5000\times100 + 15000\times100^2 + 20000\times100^3 + 10000\times100^4$$

$$1005^4 = 625 + 500{,}000 + 150{,}000{,}000 + 20{,}000{,}000{,}000 + 1{,}000{,}000{,}000{,}000$$

$$\therefore 1005^4 = \begin{array}{r} 1{,}000{,}000{,}000{,}000 \\ 20{,}000{,}000{,}000 \\ 150{,}000{,}000 \\ 500{,}000 \\ 625 \\ \hline 1{,}020{,}150{,}500{,}625 \end{array}$$

IYGB - SYNOPTIC PAPER V - QUESTION 4

SETTING UP 2 EQUATIONS, $r=5$, USING AREA $= \frac{1}{2}r^2\theta^c$ & $L = r\theta^c$

$$\Rightarrow \frac{1}{2}\times 5^2 \times \theta + \frac{1}{2}\times 5^2 \times \phi = 20$$

$$\Rightarrow \frac{25}{2}\theta + \frac{25}{2}\phi = 20$$

$$\Rightarrow 25\theta + 25\phi = 40$$

$$\Rightarrow 5\theta + 5\phi = 8$$

&

$$\overset{\frown}{AB} = \overset{\frown}{BC} + 3.5$$

$$5\theta = 5\phi + 3.5$$

SIMPLE SUBSTITUTION

- $(5\phi + 3.5) + 5\phi = 8$

  $10\phi = 4.5$

  $\phi = 0.45^c$

- $5\theta + 5\phi = 8$

  $5\theta + \phi = 1.6$

  $\theta + 0.45 = 1.6$

  $\theta = 1.15^c$

IYGB - SYNOPTIC PAPER V - QUESTION 5

USING INTEGRATION BY PARTS WITHOUT LIMITS

$$\int (\ln x)^2 \times 1 \, dx = x(\ln x)^2 - \int 2\ln x \, dx$$

$(\ln x)^2$	$2(\ln x) \times \frac{1}{x}$
x	1

ANOTHER INTEGRATION BY PARTS

$$= x(\ln x)^2 - \left[2x\ln x - \int 2 \, dx \right]$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

$\ln x$	$\frac{1}{x}$
$2x$	2

INSERTING THE INTEGRATION LIMITS

$$\int_1^e (\ln x)^2 \, dx = \left[x(\ln x)^2 - 2x\ln x + 2x \right]_1^e$$

$$= (e - \cancel{2e} + \cancel{2e}) - (0 - 0 + 2)$$

$$= e - 2$$

IYGB ~ SYNOPTIC PAPER V – QUESTION 6

REVERSE THE ORDER AS WELL AS THE TRANSFORMATIONS THEMSELVES

- "DOWN TRANSLATION BY 1 UNIT", IE "$f(x)-1$"

$$y=\frac{x^2+3x+3}{x^2+4x+5} \longmapsto y=\frac{x^2+3x+3}{x^2+4x+5}-1$$

- "REFLECTION ABOUT THE y AXIS", AS IT INVERSES ITSELF, IE $f(-x)$

$$y=\frac{x^2+3x+3}{x^2+4x+5}-1 \longmapsto y=\frac{(-x)^2+3(-x)+3}{(-x)^2+4(-x)+5}-1$$

$$y=\frac{x^2-3x+3}{x^2-4x+5}-1$$

- "TRANSLATION LEFT, BY 2 UNITS", IE $f(x+2)$

$$y=\frac{x^2-3x+3}{x^2-4x+5}-1 \longmapsto y=\frac{(x+2)^2-3(x+2)+3}{(x+2)^2-4(x+2)+5}-1$$

FINALLY TIDYING UP

$$y=\frac{x^2+4x+4-3x-6+3}{x^2+4x+4-4x-8+5}-1=\frac{x^2+x+1}{x^2+1}-1$$

$$=\frac{x^2+x+1-(x^2+1)}{x^2+1}$$

$$=\frac{x^2+x+1-x^2-1}{x^2+1}$$

$$=\frac{x}{x^2+1}$$

$$\therefore\ y=\frac{x}{x^2+1}$$

IYGB - SYNOPTIC PAPER V - QUESTION 7

PROCEED AS FOLLOWS NOTING THE SOLUTIONS MUST BE SYMMETRICAL

$$\Rightarrow \frac{1}{x} + \frac{1}{y} = \frac{8}{15}$$

$$x + y = 10$$

$$y = 10 - x$$

$$\Rightarrow \frac{y + x}{xy} = \frac{8}{15}$$

$$\Rightarrow \frac{10}{xy} = \frac{8}{15}$$

$$\Rightarrow xy = \frac{150}{8}$$

$$\Rightarrow xy = \frac{75}{4}$$

$$\Rightarrow x(10 - x) = \frac{75}{4}$$

$$\Rightarrow 10x - x^2 = \frac{75}{4}$$

$$\Rightarrow 40x - 4x^2 = 75$$

$$\Rightarrow 0 = 4x^2 - 40x + 75$$

QUADRATIC FORMULA (OR FACTORIZATION)

$$\Rightarrow (2x - 15)(2x - 5) = 0$$

$$\Rightarrow x = \begin{cases} 5/2 \\ 15/2 \end{cases} \qquad y = \begin{cases} 15/2 \\ 5/2 \end{cases}$$

$\therefore$ SOLUTIONS $\frac{5}{2}$ & $\frac{15}{2}$ EITHER ORDER

IYGB - SYNOPTIC PAPER V - QUESTION 8

FORMING STANDARD EQUATIONS BASED ON $S_n = \frac{n}{2}[2a+(n-1)d]$

$S_8 = 124$

$\frac{8}{2}[2a+7d] = 124$

$2a + 7d = 31$

$S_{20} = 910$

$\frac{20}{2}[2a+19d] = 910$

$2a + 19d = 91$

$31 - 7d = 91 - 19d$

$12d = 60$

$d = 5$

$\Rightarrow 2a + 7d = 31$

$\Rightarrow 2a + 35 = 31$

$\Rightarrow 2a = -4$

$\Rightarrow a = -2$

FINALLY USING $u_n = a + (n-1)d$ OR $u_k = a + (n-1)d$ WITH $u_k = 193$

$\Rightarrow 193 = -2 + (k-1)\times 5$

$\Rightarrow 195 = 5(k-1)$

$\Rightarrow 195 = 5k - 5$

$\Rightarrow 200 = 5k$

$\therefore k = 40$

IYGB - SYNOPTIC PAPER V - QUESTION 9

AS $x=-2$, SUBSTITUTE INTO THE EQUATION IT SATISFIES

$$\Rightarrow p(x) = k$$
$$\Rightarrow (x^2-2x-4)^2-15(x^2-2x-4) = k$$
$$\Rightarrow (4+4-4)^2-15(4+4-4) = k$$
$$\Rightarrow 16-60 = k$$
$$\Rightarrow k = -44$$

THUS WITH $k=-44$ WE HAVE

$$\Rightarrow (x^2-2x-4)^2-15(x^2-2x-4) = -44$$
$$\Rightarrow (x^2-2x-4)^2-15(x^2-2x-4)+44 = 0$$
$$\Rightarrow A^2-15A+44=0$$

WITH $A = x^2-2x-4$

$$\Rightarrow (A-11)(A-4)=0$$
$$\Rightarrow A = \begin{cases} 4 \\ 11 \end{cases}$$
$$\Rightarrow x^2-2x-4 = \begin{cases} 4 \\ 11 \end{cases}$$

SOLVE EACH QUADRATIC SEPARATELY

- $x^2-2x-4=4$

$$\Rightarrow x^2-2x-8=0$$
$$\Rightarrow (x+2)(x-4)=0$$
$$\Rightarrow x = \begin{cases} -2 \\ 4 \end{cases}$$

- $x^2-2x-4=11$

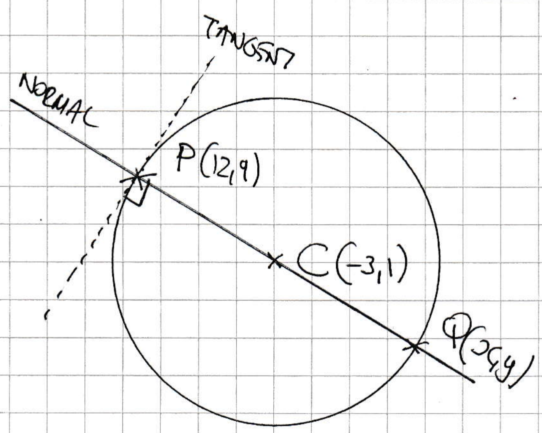
$$\Rightarrow x^2-2x-15=0$$
$$\Rightarrow (x-5)(x+3)=0$$
$$\Rightarrow x = \begin{cases} 5 \\ -3 \end{cases}$$

$\therefore$ THE OTHER 3 VALUES ARE $4, 5$ & -3

IYGB - SYNOPTIC PAPER V - QUESTION 10

a) LOOKING AT THE DIAGRAM BELOW & FILLING IN RELEVANT DETAILS

- THE NORMAL AT P MUST PASS THROUGH THE CENTRE



- GRADIENT OF NORMAL $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{1-9}{-3-12}$$

$$= \frac{-8}{-15}$$

$$= \frac{8}{15}$$

- EQUATION OF NORMAL IS GIVEN BY

$$y - y_0 = m(x - x_0)$$
$$y - 9 = \frac{8}{15}(x - 12)$$
$$15y - 135 = 8x - 96$$
$$15y - 8x - 39 = 0$$
$$8x - 15y + 39 = 0$$

b) WE COULD SOLVE SIMULTANEOUS EQUATIONS BETWEEN

$$(x+3)^2 + (y-1)^2 = 289$$
$$8x - 15y + 39 = 0$$

BUT THERE IS A SIMPLER METHOD

EITHER

$$12 \xrightarrow{-15} -3 \xrightarrow{-15} -18$$
$$9 \xrightarrow{-8} 1 \xrightarrow{-8} -7$$

P C Q

$\therefore Q(-18, -7)$

OR

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \text{"CENTRE"}$$
$$\left(\frac{x+12}{2}, \frac{y+9}{2}\right) = (-3, 1)$$
$$(x+12, y+9) = (-6, 2)$$
$$(x, y) = (-18, -7)$$

$Q(-18, -7)$

IYGB - SYNOPTIC PAPER V - QUESTION 11

FIRSTLY USING THE nth TERM

$$u_1 = 3^1 + (-2)^1 = 1 \qquad C = 1$$
$$u_2 = 3^2 + (-2)^2 = 13 \qquad D = 13$$

NOW WE HAVE

- $u_n = 3^n + (-2)^n$
- $u_{n+1} = 3^{n+1} + (-2)^{n+1} = 3 \times 3^n - 2 \times 2^n$
- $u_{n+2} = 3^{n+2} + (-2)^{n+2} = 9 \times 3^n + 4 \times 2^n$

THUS WE HAVE

$$\Rightarrow u_{n+2} = Au_{n+1} + Bu_n$$
$$\Rightarrow 9 \times 3^n + 4 \times 2^n = A[3 \times 3^n - 2 \times 2^n] + B[3^n + (-2)^n]$$
$$\Rightarrow 9 \times 3^n + 4 \times 2^n = 3A \times 3^n - 2A \times 2^n + B \times 3^n + B \times 2^n$$

EQUATING $[3^n]$ & $[2^n]$

$$\left. \begin{array}{l} 9 = 3A + B \\ 4 = -2A + B \end{array} \right\} \Rightarrow \text{SUBTRACTING } 5A = 5 \text{ \& } A = 1$$
$$\text{\& } B = 6$$

$$\therefore \quad u_{n+2} = u_{n+1} + 6u_n \quad \text{WITH } u_1 = 1,\ u_2 = 13$$

IYGB – SYNOPTIC PAPER V – QUESTION 12

a) FORMING COMPOSITIONS & SOLVE THE EQUATION

$$\Rightarrow f(g(x)) = g(f(x))$$
$$\Rightarrow f(2x+1) = g(x^2)$$
$$\Rightarrow (2x+1)^2 = 2x^2+1$$
$$\Rightarrow 4x^2+4x+1 = 2x^2+1$$
$$\Rightarrow 2x^2+4x=0$$
$$\Rightarrow 2x(x+2)=0$$

$\therefore x = 0$ or -2

b) LET $g(x) = y$

$$y = 2x+1$$
$$2x = y-1$$
$$x = \frac{1}{2}(y-1)$$

$\therefore g^{-1}(x) = \frac{1}{2}(x-1)$

c) PROCEED AS FOLLOWS

$$ghf(x) = 3-2x^2$$
$$\underbrace{g^{-1}g}\,hf(x) = g^{-1}(3-2x^2)$$

CANCEL OUT AS IDENTITY

$$h(f(x)) = \frac{1}{2}(3-2x^2-1)$$
$$h(f(x)) = \frac{1}{2}(2-2x^2)$$
$$h(x^2) = 1-x^2$$

$\therefore h(x) = 1-x$

BY INSPECTION

IYGB - SYNOPTIC PAPER V - QUESTION 13

a) **SOLVING SIMULTANEOUSLY TO FIND INTERSECTIONS**

$$\left.\begin{array}{l} y = -3x-1 \\ x^3 + y - xy = 0 \end{array}\right\} \Rightarrow x^3 + (-3x-1) - x(-3x-1) = 0$$

$$\Rightarrow x^3 - 3x - 1 + 3x^2 + x = 0$$

$$\Rightarrow \underline{\underline{x^3 + 3x^2 - 2x - 1 = 0}}$$

LET $f(x) = x^3 + 3x^2 - 2x - 1$

$$\left.\begin{array}{l} f(-0.3) = -0.157 < 0 \\ f(-0.4) = +0.216 > 0 \end{array}\right\}$$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN, THERE IS AT LEAST ONE ROOT IN THE INTERVAL

b) **REARRANGING $f(x)=0$ FOR $2x$**

$$\Rightarrow x^3 + 3x^2 - 1 = 2x$$

$$\Rightarrow x = \frac{1}{2}\left(x^3 + 3x^2 - 1\right)$$

I.E. $\underline{x_{n+1} = \frac{1}{2}\left[x_n^3 + 3x_n^2 - 1\right]}$ $(P = 1 \quad Q = 3)$

$x_1 = -0.35$

$x_2 = -0.3376875\ldots$

$x_3 = -0.3482044162\ldots$

$x_4 = -0.3392397384\ldots$

$x_5 = -0.346895065\ldots$

$x_6 = -0.3403677354\ldots$

$x_7 = -0.3459405444\ldots$

$x_8 = -0.3411879053\ldots$

$x_9 = -0.3451449232\ldots$

$x_{10} = -0.3417844861\ldots$

$x_{11} = -0.3447381044\ldots$

$x_{12} = -0.3422185$

$\therefore x_A \simeq -0.34$

IYGB - SYNOPTIC PAPER V - QUESTION 13

c) PREPARING THE "N-R" METHOD

$$f(x) = x^3 + 3x^2 - 2x - 1$$

$$f'(x) = 3x^2 + 6x - 2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n^2 - 2x_n - 1}{3x_n^2 + 6x_n - 1}, \quad x_1 = 0.8$$

$$x_2 \approx 0.8 - \frac{0.8^3 + 3\times0.8^2 - 2\times0.8 - 1}{3\times0.8^2 + 6\times0.8 - 2} \approx 0.83559\ldots$$

$$x_3 \approx 0.834245\ldots$$

$$\therefore \ x_B \approx 0.834$$

IYGB - SYNOPTIC PAPER V - QUESTION 14

EXTRACTING THE LOGS

$$\Rightarrow 2\log_2 x + \log_2(x-1) - \log_2(5x+4) = 1$$

$$\Rightarrow \log_2 x^2 + \log_2(x-1) - \log_2(5x+4) = \log_2 2$$

$$\Rightarrow \log_2\left[\frac{x^2(x-1)}{5x+4}\right] = \log_2\left[2\right]$$

$$\Rightarrow \frac{x^2(x-1)}{5x+4} = 2$$

$$\Rightarrow x^3 - x^2 = 10x + 8$$

$$\Rightarrow x^3 - x^2 - 10x - 8 = 0$$

BY INSPECTION $x=-1$ IS AN OBVIOUS SOLUTION OF THE CUBIC

$$\Rightarrow x^2(x+1) - 2x(x+1) - 8(x-1) = 0$$

$$\Rightarrow (x+1)(x^2 - 2x - 8) = 0$$

$$\Rightarrow (x+1)(x+2)(x-4) = 0$$

LONG DIVISION METHODS ARE PROBABLY BETTER!

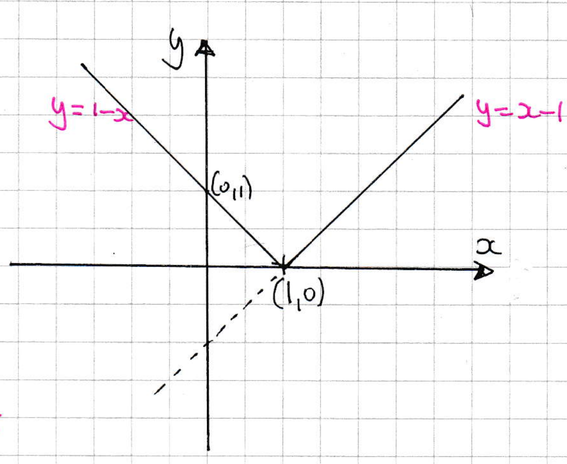
$$x = \cancel{-1},\ \cancel{-2},\ 4$$

log arguments have to be positive

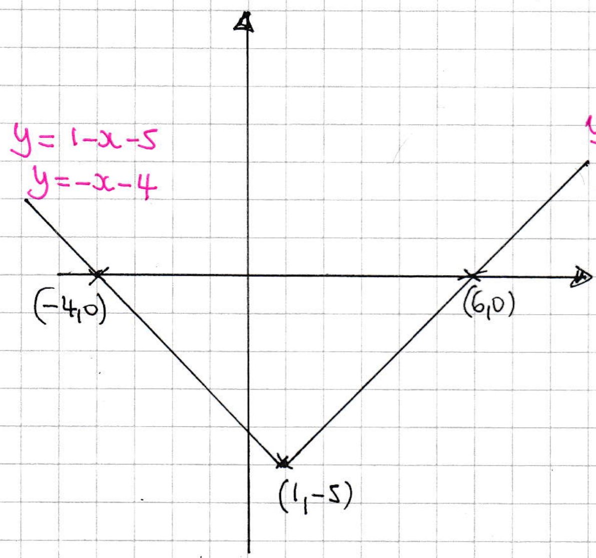
IYGB - SYNOPTIC PAPER V - QUESTION 15

BEST TO START WITH A SKETCH

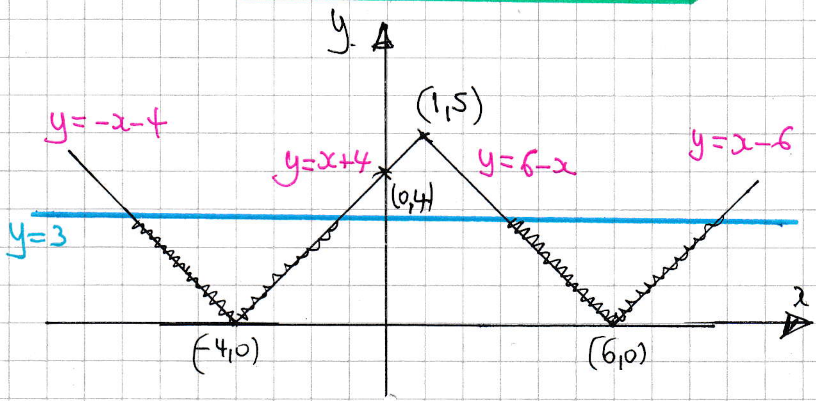
$y = |x-1|$ (SHOWN OPPOSITE)



NEXT TRANSLATING DOWN BY 5



FINALLY "MODING" THE GRAPH



INTERCEPTING WITH $y=3$ AND LOOKING AT THE "CURLY" BITS

- $-x-4=3$
  $-7=x$
- $x+4=3$
  $x=-1$
- $6-x=3$
  $3=x$
- $x-6=3$
  $x=9$

$\therefore \quad -7 < x < -1$ OR $3 < x < 9$

IYGB - SYNOPTIC PAPER V - QUESTION 16

a) USING THE SUBSTITUTION (GIVEN CLUE)

$$\int_{\sqrt{6}}^{\sqrt{8}} \frac{x^3}{x^2-4}\,dx = \int_2^4 \frac{x^3}{u}\left(\frac{du}{2x}\right)$$

$$= \int_2^4 \frac{x^2}{2u}\,du$$

$$= \int_2^4 \frac{u+4}{2u}\,du$$

SPLIT THE FRACTION

$$= \int_2^4 \frac{1}{2} + \frac{2}{u}\,du$$

$$= \left[\frac{1}{2}u + 2\ln|u|\right]_2^4$$

$$= (2+2\ln 4) - (1+2\ln 2)$$

$$= 2 + 2\ln 4 - 1 - \ln 4$$

$$= 1 + \ln 4$$

AS REQUIRED

$u = x^2-4$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$x=\sqrt{6} \mapsto u=2$

$x=\sqrt{8} \mapsto u=4$

$x^2 = u+4$

b) STANDARD PARTIAL FRACTIONS

$$\Rightarrow \frac{x^3}{(x-2)(x+2)} \equiv Ax+B+\frac{C}{x-2}+\frac{D}{x+2}$$

$$\Rightarrow x^3 \equiv (Ax+B)(x-2)(x+2) + C(x+2) + D(x-2)$$

$$\Rightarrow x^3 \equiv (Ax+B)(x^2-4) + Cx + 2C + Dx - 2D$$

$$\Rightarrow x^3 \equiv Ax^3 + Bx^2 - 4Ax - 4B + Cx + 2C + Dx - 2D$$

$$\Rightarrow x^3 \equiv Ax^3 + Bx^2 + (C-4A+D)x + (2C-4B-2D)$$

$\therefore$ $A=1$ $\quad B=0$

IYGB - SYNOPTIC PAPER V - QUESTION 16

NOW WE HAVE

$[x^1]: \quad C - 4 + D = 0$
$[x^0]: \quad 2C - 2D = 0$
$\Rightarrow \quad C + D = 4, \quad C = D, \quad C = D = 2$

$\therefore \quad A = 1, \; B = 0, \; C = 2, \; D = 2$

c) USING THE PARTIAL FRACTIONS METHOD

$$\int_{\sqrt{6}}^{\sqrt{8}} f(x)\,dx = \int_{\sqrt{6}}^{\sqrt{8}} x + \frac{2}{x-2} + \frac{2}{x+2}\,dx$$

$$= \left[\frac{1}{2}x^2 + 2\ln|x-2| + 2\ln|x+2|\right]_{\sqrt{6}}^{\sqrt{8}}$$

$$= \left[\frac{1}{2}x^2 + 2\left(\ln(x+2) + \ln(x-2)\right)\right]_{\sqrt{6}}^{\sqrt{8}}$$

$$= \left[\frac{1}{2}x^2 + 2\ln\left[(x+2)(x-2)\right]\right]_{\sqrt{6}}^{\sqrt{8}}$$

$$= \left[\frac{1}{2}x^2 + 2\ln(x^2-4)\right]_{\sqrt{6}}^{\sqrt{8}}$$

$$= \left[4 + 2\ln 4\right] - \left[3 + 2\ln 2\right]$$

$$= 4 + 2\ln 4 - 3 - \ln 4$$

$$= 1 + \ln 4$$

AS BEFORE

IYGB - SYNOPTIC PAPER V - QUESTION 17

START WITH THE DEFINITION OF THE DERIVATIVE

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

HERE $f(x) = 2x^2 + 3x + C$

$$f'(x) = \lim_{h \to 0}\left[\frac{2(x+h)^2 + 3(x+h) + \cancel{C} - (2x^2 + 3x + \cancel{C})}{h}\right]$$

$$f'(x) = \lim_{h \to 0}\left[\frac{2(x+h)^2 + 3(x+h) - 2x^2 - 3x}{h}\right]$$

NOW LET $x = 1$ & HOPE WE 'MATCH THE -5"

$$f'(1) = \lim_{h \to 0}\left[\frac{2(1+h)^2 + 3(1+h) - 2\times1^2 - 3\times1}{h}\right]$$

$$f'(1) = \lim_{h \to 0}\left[\frac{2(1+h)^2 + 3(1+h) - 5}{h}\right]$$

THUS THE FUNCTION IS INDEED THE ONE QUOTED EARLIER

$$f(x) = 2x^2 + 3x + C$$

$$f'(x) = 4x + 3$$

$$f'(1) = 7$$

$$\therefore \lim_{h \to 0}\left[\frac{(1+h)^2 + 3(1+h) - 5}{h}\right] = 7$$

IYGB - SYNOPTIC PAPER V - QUESTION 18

a) USING TRIGONOMETRIC IDENTITIES ON THE L.H.S.

$$\sin(3\theta) \equiv \sin(2\theta+\theta)$$
$$\equiv \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$$
$$\equiv (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta$$
$$\equiv 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$$
$$\equiv 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$$
$$\equiv 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$
$$\equiv 3\sin\theta - 4\sin^3\theta$$

AS REQUIRED

b) PROCEED AS FOLLOWS

$$\Rightarrow \arcsin x = 3\arcsin\tfrac{1}{3}$$
$$\Rightarrow \sin(\arcsin x) = \sin\left[3\arcsin\tfrac{1}{3}\right]$$
$$\Rightarrow x = \sin 3\theta$$

WITH $\theta = \arcsin\frac{1}{3}$

$\sin\theta = \frac{1}{3}$

USING PART (a)

$$\Rightarrow x = 3\sin\theta - 4\sin^3\theta$$
$$\Rightarrow x = 3\times\tfrac{1}{3} - 4\left(\tfrac{1}{3}\right)^3$$
$$\Rightarrow x = 1 - \tfrac{4}{27}$$
$$\Rightarrow x = \tfrac{23}{27}$$

IYGB - SYNOPTIC PAPER V - QUESTION 19

REWRITE AND TAKE NATURAL LOGS

$$\Rightarrow f(x) = \frac{\sqrt{1+6\sin^2 x}}{(1+\tan x)^2} = \frac{(1+6\sin^2 x)^{\frac{1}{2}}}{(1+\tan x)^2}$$

$$\Rightarrow \ln[f(x)] = \ln\left[\frac{(1+6\sin^2 x)^{\frac{1}{2}}}{(1+\tan x)^2}\right]$$

$$\Rightarrow \ln[f(x)] = \ln(1+6\sin^2 x)^{\frac{1}{2}} - \ln(1+\tan x)^2$$

$$\Rightarrow \ln[f(x)] = \frac{1}{2}\ln(1+6\sin^2 x) - 2\ln(1+\tan x)$$

NOW LET US NOTE THAT

$$f\left(\tfrac{\pi}{4}\right) = \frac{\sqrt{1+3}}{4} = \frac{1}{2}$$

DIFFERENTIATE $f(x)$ WITH RESPECT TO x

$$\frac{1}{f(x)}f'(x) = \frac{1}{2}\times\frac{1}{1+6\sin^2 x}\times 12\sin x\cos x - 2\times\frac{1}{1+\tan x}\times\sec^2 x$$

$$\frac{1}{\frac{1}{2}}f'(\pi/4) = \frac{1}{2}\times\frac{1}{1+3}\times 6 - 2\times\frac{1}{2}\times 2$$

$$2f'(\pi/4) = \frac{3}{4} - 2$$

$$2f'(\pi/4) = -\frac{5}{4}$$

$$f'(\pi/4) = -\frac{5}{8}$$

IYGB - SYNOPTIC PAPER V - QUESTION 20

$$\frac{dy}{dx} = \frac{k(9-x)}{y} \quad \text{SUBJECT TO } y=\frac{1}{2},\ \frac{dy}{dx}=2 \text{ AT } x=1$$

SUBSTITUTE THE GIVEN CONDITIONS INTO THE O.D.E TO OBTAIN k

$$\Rightarrow \frac{1}{2} = \frac{k(9-1)}{2}$$
$$\Rightarrow 1 = 8k$$
$$\Rightarrow k = \frac{1}{8}$$

SOLVE THE O.D.E BY SEPARATION OF VARIABLES

$$\Rightarrow y\,dy = k(9-x)\,dx$$
$$\Rightarrow \int y\,dy = \int k(9-x)\,dx$$
$$\Rightarrow \frac{1}{2}y^2 = -k(9-x)^2 \times \frac{1}{2} + C$$
$$\Rightarrow y^2 = C - k(9-x)^2$$

APPLY $x=1$, $y=\frac{1}{2}$

$$\Rightarrow \frac{1}{4} = C - \frac{1}{8}(9-1)^2$$
$$\Rightarrow \frac{1}{4} = C - 8$$
$$\Rightarrow C = 8 + \frac{1}{4} = \frac{33}{4}$$

$$\therefore \quad y^2 = \frac{33}{4} - \frac{1}{8}(9-x)^2$$

NOW SETTING $\frac{dy}{dx} = \frac{1}{5}$ INTO THE O.D.E

$$\Rightarrow \frac{1}{5} = \frac{1}{8}\frac{(9-x)}{y}$$
$$\Rightarrow 8y = 5(9-x)$$
$$\Rightarrow y = \frac{5}{8}(9-x)$$

IYGB - SYNOPTIC PAPER V - QUESTION 20

NOW WHEN $\frac{dy}{dx} = \frac{1}{5}$ $\quad y = \frac{5}{8}(9-x)$

$$y^2 = \frac{25}{64}(9-x)^2$$

THUS WE NOW HAVE

$$\left.\begin{aligned} y^2 &= \frac{33}{4} - \frac{1}{8}(9-x)^2 \\ y^2 &= \frac{25}{64}(9-x)^2 \end{aligned}\right\} \Longrightarrow \text{COMBINING}$$

$$\Longrightarrow \frac{33}{4} - \frac{1}{8}(9-x)^2 = \frac{25}{64}(9-x)^2 \quad \times 64$$

$$\Longrightarrow 33 \times 16 - 8(9-x)^2 = 25(9-x)^2$$

$$\Longrightarrow 33 \times 16 = 33(9-x)^2$$

$$\Longrightarrow 16 = (9-x)^2$$

$$\Longrightarrow 9-x = \begin{cases} 4 \\ -4 \end{cases}$$

$$\Longrightarrow x-9 = \begin{cases} -4 \\ 4 \end{cases}$$

$$\Longrightarrow x = \begin{cases} 5 \\ 13 \end{cases}$$

IYGB - SYNOPTIC PAPER V - QUESTION 21

a) NOTING BY INSPECTION THAT $A\hat{M}N$ IS RIGHT ANGLED AS A PYTHAGOREAN TRIPLE (3:4:5) OR (6:8:10), IE $|AN| = 10$

$\Rightarrow P = 24$

$\Rightarrow |AB| + |BC| = 12$

$\Rightarrow 8\cos\theta + (\underbrace{8\sin\theta}_{|BM|} + \underbrace{6\cos\theta}_{|MC|}) = 12$

& $N\hat{M}C = \theta$

$\Rightarrow 8\sin\theta + 14\cos\theta = 12$

$\Rightarrow 4\sin\theta + 7\cos\theta = 6$

OBTAIN R- TRANSFORMATION IN THE LHS, NOTING $R = \sqrt{4^2+7^2} = \sqrt{65}$

$4\sin\theta + 7\cos\theta \equiv R\sin(\theta + \alpha)$

$4\sin\theta + 7\cos\theta \equiv \sqrt{65}\sin\theta\cos\alpha + \sqrt{65}\cos\theta\sin\alpha$

$\sqrt{65}\cos\alpha = 4$

$\cos\alpha = \frac{4}{\sqrt{65}}$

$\alpha \simeq 60.255...$

RETURNING TO THE EQUATION

$4\sin\theta + 7\cos\theta = 6$

$\sqrt{65}\sin(\theta + 60.255...) = 6$

$\sin(\theta + 60.255...) = \frac{6}{\sqrt{65}}$

$\arcsin\left(\frac{6}{\sqrt{65}}\right) = 48.09...$

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$$\Rightarrow \begin{cases} \theta + 60.255... = 48.09 \pm 360n \\ \theta + 60.255... = 131.91 \pm 360n \end{cases} \qquad n = 0, 1, 2, 3, ...$$

$$\Rightarrow \begin{cases} \theta = -12.16... \pm 360n \\ \theta = 71.66 \pm 360n \end{cases}$$

$$\therefore \theta = \begin{cases} \cancel{347.8°} \\ 71.7° \end{cases}$$

θ MUST BE ACUTE

b) AREA OF $\hat{ADN}$ NEXT

$$\text{AREA} = \frac{1}{2}|AD||DN|$$

$$= \frac{1}{2}[|BM| + |MC|][|AB| - |NC|]$$

$$= \frac{1}{2}[8\sin\theta + 6\cos\theta][8\cos\theta - 6\sin\theta]$$

$$= \frac{1}{2}[64\sin\theta\cos\theta - 48\sin^2\theta + 48\cos^2\theta - 36\sin\theta\cos\theta]$$

$$= \frac{1}{2}[28\sin\theta\cos\theta - 48(\cos^2\theta - \sin^2\theta)]$$

$$= 14\sin\theta\cos\theta - 24(\cos^2\theta - \sin^2\theta)$$

$$= 7\sin 2\theta - 24\cos 2\theta$$

BY R - TRANSFORMATION

$$= R\sin(2\theta + \beta)$$

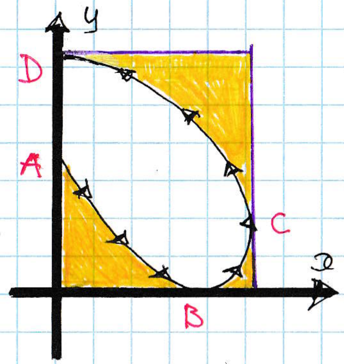
$$\sqrt{7^2 + 24^2} = 25$$

$\therefore$ AREA MAX IS 25

IYGB - SYNOPTIC PAPER V - QUESTION 22

START BY "TRACING" THE CURVE

- "MAX x" $= 1 \implies \theta = \frac{\pi}{4}$
- "MAX y" $= 2 \implies \sin 3\theta = -1$
  $3\theta = \frac{3\pi}{2}$
  $\theta = \frac{\pi}{2}$
- $\theta = 0$ YIELDS $(0,1)$
- $x = 0$ YIELDS $\theta = 0, \frac{\pi}{2}$
- $y = 0$ YIELDS $\theta = \frac{\pi}{6}$



$x = \sin 2\theta$
$y = 1 - \sin 3\theta$
$0 \leq \theta \leq \frac{\pi}{2}$

HENCE BY INSPECTION

	A	B	C	D
θ	0	$\pi/6$	$\pi/4$	$\pi/2$
x	0	$\sqrt{3}/2$	1	0
y	1	0	$1-\frac{\sqrt{2}}{2}$	2

NEXT WE FIND THE AREA ENCLOSED BY THE LOOP IN THE FIRST QUADRANT

$$\text{AREA} = \int_{\theta_1}^{\theta_2} y(\theta)\,\frac{dx}{d\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} (1 - \sin 3\theta)(2\cos 2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\cos 2\theta - 2\sin 3\theta \cos 2\theta\,d\theta$$

USING TRIGONOMETRIC IDENTITIES

$\sin(3\theta + 2\theta) = \sin 5\theta = \sin 3\theta \cos 2\theta + \cos 3\theta \sin 2\theta$
$\sin(3\theta - 2\theta) = \sin\theta = \sin 3\theta \cos 2\theta - \cos 3\theta \cos 2\theta$

ADDING YIELDS

$$\boxed{\sin 5\theta + \sin\theta = 2\sin 3\theta \cos 2\theta}$$

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RESUMING THE INTEGRATION

$$\ldots = \int_0^{\frac{\pi}{2}} 2\cos 2\theta - (\sin\theta + \sin 5\theta)\, d\theta$$

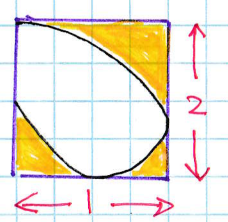
$$= \int_0^{\frac{\pi}{2}} 2\cos 2\theta - \sin\theta - \sin 5\theta \quad d\theta$$

$$= \left[\sin 2\theta + \cos\theta + \frac{1}{5}\cos 5\theta\right]_0^{\frac{\pi}{2}}$$

$$= (0+0+0) - \left(0+1+\frac{1}{5}\right)$$

$$= \frac{6}{5}$$ ← AREA OF THE "LOOP"

NEXT THE RECTANGLE



HENCE THE REQUIRED AREA OF THE YELLOW GLASS

$$\text{YELLOW GLASS} = 4 \times \left(2 - \frac{6}{5}\right)$$

$$= \frac{16}{5}$$