IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper W

Difficulty Rating: 4.0225/0.7080

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 22 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

A recurrence relation is defined for $n \geq 1$ by

$$a_{n+1} = 7a_n - n^3 - 3, \quad a_1 = 1.$$

a) Find the value of a_4. **(3)**

b) Evaluate the sum

$$\sum_{r=1}^{5} a_r .$$ **(2)**

Question 2

$$f(x) \equiv x^4 + 2x^3 + x^2 - 4, \ x \in \mathbb{R}.$$

a) Use the factor theorem to show that $(x+2)$ is a factor of $f(x)$. **(2)**

b) Express $f(x)$ as the product of a linear factor and a cubic factor. **(2)**

c) Find another linear factor of $f(x)$. **(2)**

d) Express $f(x)$ as the product of two linear factors and a quadratic factor. **(1)**

e) Show that the equation $f(x) = 0$ has exactly two solutions. **(2)**

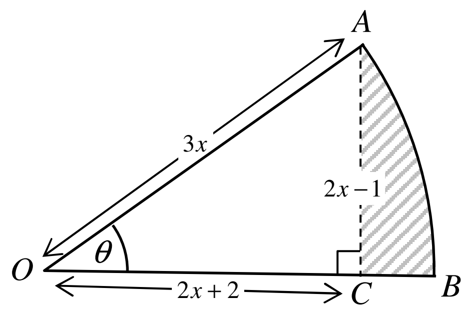
Question 3

Solve each of the following equations.

a) $x = |3x+2| - 4$. **(4)**

b) $x^2 + 1 = |2x-4|$. **(4)**

Question 4



The figure above shows a circular sector OAB of radius $3x$ cm, subtending an angle θ radians at O.

The line AC is perpendicular to OB and has length $(2x-1)$ cm.

The length of OC is $(2x+2)$ cm.

a) Show that $x=5$. **(3)**

b) Find the area of the shaded region ACB. **(5)**

Question 5

$$f(x)=\frac{169}{8}-2\left(x+\frac{7}{4}\right)^2,\ x\in\mathbb{R}.$$

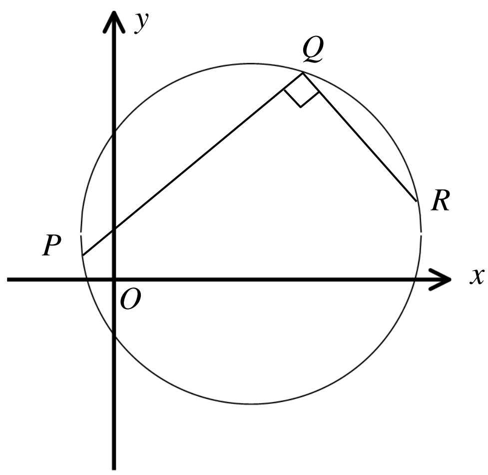
a) State the coordinates of the maximum point of $f(x)$. **(2)**

b) Express $f(x)$ in the form ax^2+bx+c, where a, b and c are integers. **(3)**

c) Solve the equation $f(x)=0$. **(2)**

d) Sketch the graph of $f(x)$, indicating clearly the coordinates of the points where the graph of $f(x)$ meets the coordinate axes. **(3)**

Question 6



A circle passes through the points with coordinates $P(-2,1)$, $Q(14,13)$ and $R(20,k)$, where k is a constant.

Given that $\measuredangle PQR = 90°$, determine an equation for the circle. **(6)**

Question 7

The function $f(x)$ is given by

$$f(x) = 2x^2 + 3, \; x \in \mathbb{R}, \;\; x \le 0.$$

a) Sketch the graph of $f(x)$. **(2)**

b) Find $f^{-1}(x)$ in its simplest form. **(3)**

c) Find the domain and range of $f^{-1}(x)$. **(2)**

d) Solve the equation

$$f^{-1}(x) = -3.$$ **(2)**

Question 8

$$f(x) \equiv \frac{4x(9x-10)}{(2-x)(2-3x)^2}, \quad x \in \mathbb{R}, \quad |x| < \frac{2}{3}, \quad x \neq 0.$$

a) Find the values of the constants A, B and C given that

$$f(x) \equiv \frac{A}{2-x} + \frac{B}{2-3x} + \frac{C}{(2-3x)^2}. \qquad (4)$$

b) Hence, or otherwise, find the binomial series expansion of $f(x)$, up and including the term in x^2. **(6)**

The equation $f(x) = -0.63$ is known to have a positive solution which is further known to be numerically small.

c) Use part **(b)** to find this solution. **(3)**

Question 9

The straight line with equation

$$y = k(4x-17),$$

does **not** intersect with the quadratic with equation

$$y = 13 - 8x - x^2.$$

Find the range of possible values of k. **(7)**

Question 10

Solve the following simultaneous equations **without** using a calculator

$$8^y = 4^{2x+1}$$
$$27^{2y} = 9^{x-3} \qquad (4)$$

Question 11

Use the substitution $u = \ln x$ to show that

$$\int 3^{\ln x}\, dx = \frac{x\left(3^{\ln x}\right)}{1+\ln 3} + \text{constant} . \qquad \textbf{(7)}$$

Question 12

An implicit relationship between x and y is given below, in terms of a constant A.

$$y\sec^2 x = A + 2\ln(\sec x),\ 0 \le x < \frac{\pi}{2} .$$

Given that $y = 2$ at $x = \frac{\pi}{3}$, **show clearly** that when $x = \frac{1}{6}\pi$

$$y = \tfrac{3}{4}(8 - \ln 3) . \qquad \textbf{(7)}$$

Question 13

It is given that

$$\sin^2\left(\theta + \frac{\pi}{4}\right) - \sin^2\left(\theta - \frac{\pi}{4}\right) \equiv \sin 2\theta .$$

a) Prove the validity of the above trigonometric identity. **(4)**

b) Hence, or otherwise, show that …

i. … $\sin\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta + \frac{\pi}{4}\right) - \sin\left(\theta - \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right) \equiv \cos 2\theta$. **(4)**

ii. … $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right) = \frac{1}{2}$. **(2)**

Question 14

Consider a sequence of geometric transformations T_1, T_2 and T_3 which map the graph of the curve with equation $y_1 = \frac{1}{x}$ onto the graph of y_2.

T_1 : reflection in the x axis.

T_2 : translation in the negative x direction by 2 units.

T_3 : translation in the positive y direction by 2 units.

a) Show that the equation of y_2 is given by $y_2 = \frac{2x+3}{x+2}$, $x \neq -2$. **(5)**

b) Sketch the graph of y_2.

You must clearly indicate the equations of any asymptotes and coordinates of any intersections with the coordinate axes. **(5)**

c) Solve the equation

$$\frac{2x+3}{x+2} = 2 + \frac{2}{x-1}.$$ **(3)**

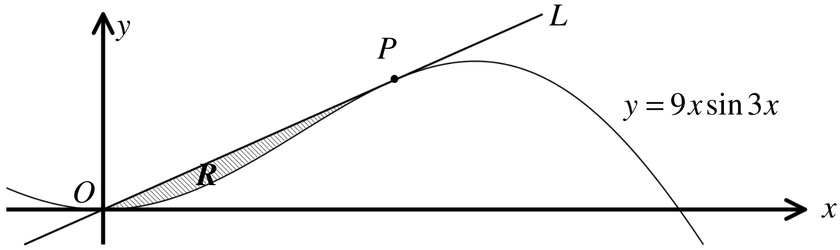
Question 15

$$y = (x+2)^2 e^{1-x}, \; x \in \mathbb{R}$$

Show clearly that

$$(x+2)^2 \frac{d^2y}{dx^2} + x(x+2)\frac{dy}{dx} + 2y = 0.$$ **(7)**

Question 16



The figure above shows the graph of the curve C with equation

$$y = 9x\sin 3x, \ x \in \mathbb{R}.$$

The straight line L is the tangent to C at the point P, whose x coordinate is $\frac{\pi}{6}$.

a) Show that L passes through the origin O. **(6)**

The finite region R bounded by C and L.

b) Show further that the area of R is

$$\frac{1}{8}\left(\pi^2 - 8\right).$$ **(7)**

Question 17

Solve the following simultaneous equations

$$\arctan x + \arctan y = \arctan 8$$

$$x + y = 2.$$ **(7)**

Question 18

A curve C is given by the parametric equations

$$x = 2\cos 2t, \quad y = 5\sin t \;, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}.$$

(7)

The point $P\left(1, \frac{5}{2}\right)$ lies on C.

a) Find the value of the gradient at P, and hence, show that an equation of the normal to C at P is

$$8x - 10y + 17 = 0.$$

The normal at P meets C again at the point Q.

b) Show that the y coordinate of Q is $-\frac{165}{16}$. (7)

Question 19

Water is leaking from a hole at the **side** of a water tank.

The tank has a height of 3 m and is initially full. It is thought that while the tank is leaking, the height, H m, of the water in the tank at time t hours, is governed by the differential equation

$$\frac{dH}{dt} = -k\,e^{-0.1t},$$

where k is a positive constant.

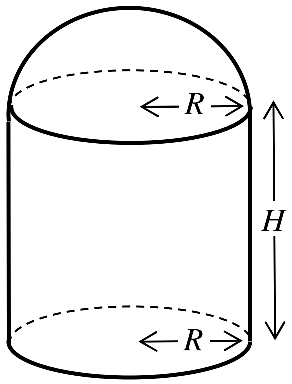
The height of the water drops to 2 metres after 10 hours.

Find in **exact simplified** form ...

a) ... an expression for H in terms of t. (9)

b) ... the height of the hole, measured from the ground. (2)

Question 20



The figure above shows a hollow container consisting of a right circular cylinder of radius R and of height H joined to a hemisphere of radius R.

The cylinder is open on one of the circular ends and the hemisphere is also open on its circular base. The cylinder is joined to the hemisphere at their open ends so that the resulting object is completely sealed.

Given that volume of the container is V, show the surface area of the container is minimised when $R = H$, and hence show further that this minimum surface area is

$$\sqrt[3]{45\pi V^2}\ .$$

(11)

Question 21

Liquid is pouring into a container at the constant rate of $2\pi\ \text{cm}^3\text{s}^{-1}$.

The container initially contains some liquid and when the height of the liquid in the container is h cm the volume of the liquid, V cm^3, is given by

$$V = \pi\left(h^2 + 5h - 16\right), \quad h \geq 6\ .$$

Determine the rate at which the height of the liquid in the container is rising 30 seconds after the liquid started pouring in.

(10)

Question 22

A convergent geometric progression has positive first term and positive common ratio.

Use differential calculus to show that the sum to infinity of the progression is at **least** four times as large as its second term.

(**11**)