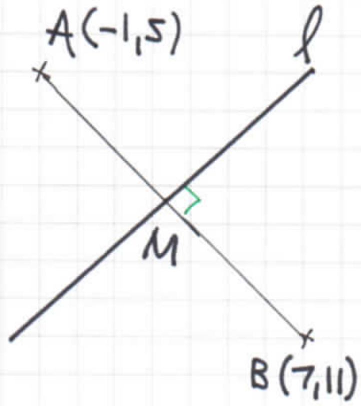
IYGB - SYNOPTIC PAPER A - QUESTION 1

a) MIDPOINT OF AB



$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = M\left(\frac{-1+7}{2}, \frac{5+11}{2}\right)$$

$M(3,8)$

GRADIENT OF AB

$$\frac{y_2-y_1}{x_2-x_1} = \frac{11-5}{7-(-1)} = \frac{6}{8} = \frac{3}{4}$$

GRADIENT OF l WILL BE $-\frac{4}{3}$

EQUATION OF THE PERPENDICULAR BISECTOR OF AB

$$\Rightarrow y - y_0 = m(x - x_0)$$
$$\Rightarrow y - 8 = -\frac{4}{3}(x-3)$$
$$\Rightarrow 3y - 24 = -4x + 12$$
$$\Rightarrow 3y + 4x = 36$$

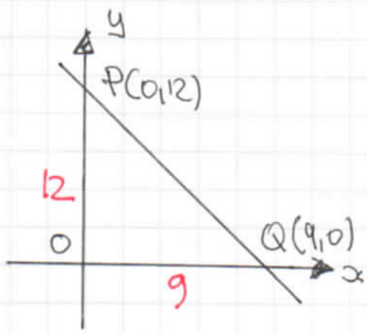
AS REQUIRED

b) DRAWING A "RELATIVE SCALE" DIAGRAM

$4x + 3y = 36$

when $x=0 \Rightarrow y = 12$

when $y=0 \Rightarrow x = 9$



REQUIRED AREA $= \frac{1}{2}|OP||OQ|$

$= \frac{1}{2} \times 12 \times 9$

$= 54$

IYGB - SYNOPTIC PAPER A - QUESTION 2

$$f(x) = x^3 + x^2 - x + k, \quad x \in \mathbb{R}$$

$(x-k)$ IS A FACTOR OF $f(x)$

$$f(k) = 0 \implies k^3 + k^2 - k + k = 0$$

$$\implies k^3 + k^2 = 0$$

$$\implies k^2(k+1) = 0$$

$$\implies k = \begin{cases} 0 \\ -1 \end{cases}$$

IYGB - SYNOPTIC PAPER A - QUESTION 3

a) I) $\sqrt{98} + \sqrt{2} = \sqrt{49}\sqrt{2} + \sqrt{2}$

$= 7\sqrt{2} + \sqrt{2}$

$= 8\sqrt{2}$ //

II) $(\sqrt{2}+3)(2-3\sqrt{2}) = 2\sqrt{2} - 3\sqrt{2}\sqrt{2} + 3\times 2 - 9\sqrt{2}$

$= 2\sqrt{2} - 6 + 6 - 9\sqrt{2}$

$= -7\sqrt{2}$ //

b) $\dfrac{27^t}{3^{t-1}} = 3\sqrt{3}$

$\Rightarrow \dfrac{(3^3)^t}{3^{t-1}} = 3^1 \times 3^{\frac{1}{2}}$

$\Rightarrow \dfrac{3^{3t}}{3^{t-1}} = 3^{\frac{3}{2}}$

$\Rightarrow 3^{3t-(t-1)} = 3^{\frac{3}{2}}$

$\Rightarrow 3^{2t+1} = 3^{\frac{3}{2}}$

$\Rightarrow 2t+1 = \frac{3}{2}$

$\Rightarrow 2t = \frac{1}{2}$

$\Rightarrow t = \frac{1}{4}$ //

VARIATION

$\Rightarrow 3^{3t} = 3^{t-1} \times 3^{\frac{3}{2}}$

$\Rightarrow 3^{3t} = 3^{t+\frac{1}{2}}$

$\Rightarrow 3t = t + \frac{1}{2}$

$\Rightarrow 2t = \frac{1}{2}$

$\Rightarrow t = \frac{1}{4}$ //

IYGB - SYNOPTIC PAPER A - QUESTION 4

$$\boxed{12 - 2|2x-3| \geqslant 7}$$

$$\Rightarrow -2|2x-3| \geqslant -5$$

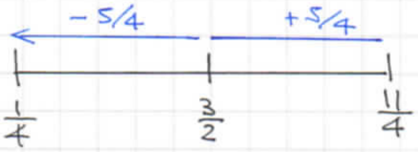
$$\Rightarrow |2x-3| \leqslant \frac{5}{2}$$

$$\Rightarrow 2\left|x-\frac{3}{2}\right| \leqslant \frac{5}{2}$$

$$\Rightarrow \left|x-\frac{3}{2}\right| \leqslant \frac{5}{4}$$

THE STATEMENT ABOVE READS

"THE DIFFERENCE OF x FROM $\frac{3}{2}$ IS LESS THAN $\frac{5}{4}$"



$$\therefore \frac{1}{4} \leqslant x \leqslant \frac{11}{4}$$

ALTERNATIVE - SOLVE AN EQUATION INSTEAD

$$\Rightarrow 12 - 2|2x-3| = 7$$

$$\Rightarrow 5 = 2|2x-3|$$

$$\Rightarrow \frac{5}{2} = |2x-3|$$

$$\Rightarrow \begin{cases} 2x-3 = \frac{5}{2} \\ 2x-3 = -\frac{5}{2} \end{cases}$$

$$\Rightarrow \begin{cases} 2x = \frac{11}{2} \\ 2x = \frac{1}{2} \end{cases}$$

$$\Rightarrow x = \begin{cases} 11/4 \\ 1/4 \end{cases}$$

THE SOLUTION INTERVAL IS EITHER BETWEEN $\frac{1}{4}$ & $\frac{11}{4}$ OR OUTSIDE

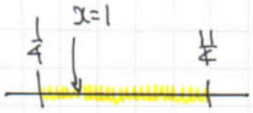
TRY A NUMBER, SAY $x=1$, IN THE INITIAL INEQUALITY

$$12 - 2|2\times1-3| \geqslant 7$$

$$12 - 2\times1 \geqslant 7$$

$$10 \geqslant 7$$

IF $x=1$ WORKS



$$\therefore \frac{1}{4} \leqslant x \leqslant \frac{11}{4}$$

IYGB − SYNOPTIC PAPER A − QUESTION 5

- START BY FINDING THE CO-ORDINATES OF B

$$\left.\begin{array}{l} y = x+3 \\ y = 9+2x-x^2 \end{array}\right\} \Rightarrow x+3 = 9+2x-x^2$$

$$\Rightarrow x^2 - x - 6 = 0$$

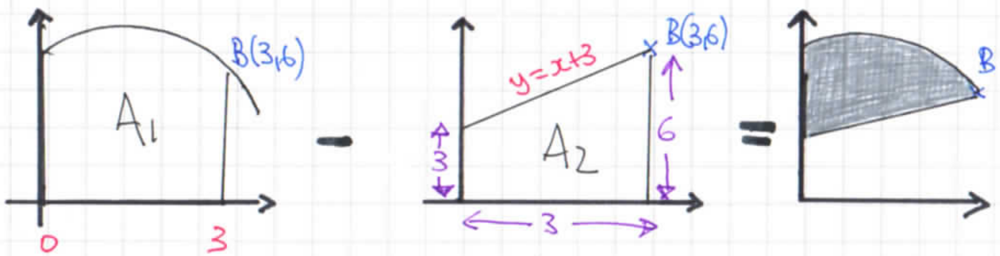
$$\Rightarrow (x-3)(x+2) = 0$$

$$\Rightarrow x = \begin{cases} \cancel{-2} \leftarrow A \\ 3 \end{cases}$$

$$\Rightarrow y = 3+3 = 6$$

$$\therefore B(3,6)$$

- THE REQUIRED AREA CAN BE FOUND AS FOLLOWS



↑ BY INTEGRATION

↑ TRAPEZIUM

$$\frac{3+6}{2} \times 3 = \frac{27}{2} = 13.5$$

- $A_1 = \int_0^3 9+2x-x^2 \, dx$

$$A_1 = \left[9x + x^2 - \frac{1}{3}x^3\right]_0^3$$

$$A_1 = (27+9-9) - (0)$$

$$A_1 = 27$$

- HENCE THE REQUIRED AREA IS

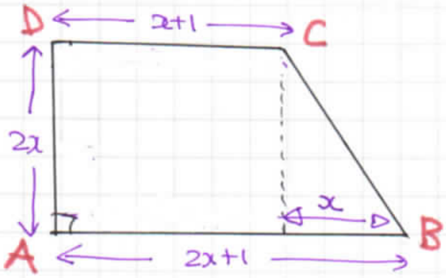
$$27 - \frac{27}{2} = 27 - 13.5$$

$$= 13.5$$

AS REQUIRED

IYGB - SYNOPTIC PAPER A - QUESTION 6

FORM AN EQUATION BASED ON THE AREA OF A TRAPEZIUM



$$\Rightarrow \frac{(x+1)+(2x+1)}{\cancel{2}} \times \cancel{2}x = 16$$

$$\Rightarrow (3x+2)x = 16$$

$$\Rightarrow 3x^2 + 2x - 16 = 0$$

$$\Rightarrow (3x+8)(x-2)$$

$$\Rightarrow x = \begin{cases} 2 \\ \cancel{-\frac{8}{3}} \end{cases} \quad x > 0$$

FINALLY BY PYTHAGORAS ON THE RIGHT ANGLED TRIANGLE

$$(2x)^2 + x^2 = |BC|^2$$

$$|BC|^2 = 5x^2$$

$$|BC| = +\sqrt{5}x$$

$$|BC| = \sqrt{5} \times 2$$

$\therefore$ THE REQUIRED LENGTH IS

$$\sqrt{5}$$

IYGB - SYNOPTIC PAPER A - QUESTION 7

a) EXPAND USING THE STANDARD FORMULA

$$f(x) = \sqrt{1+\tfrac{1}{8}x} = \left(1+\tfrac{1}{8}x\right)^{\frac{1}{2}}$$

$$f(x) = 1 + \frac{\frac{1}{2}}{1}\left(\tfrac{1}{8}x\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\times 2}\left(\tfrac{1}{8}x\right)^2 + O(x^3)$$

$$f(x) = 1 + \frac{1}{16}x - \frac{1}{512}x^2 + O(x^3)$$

b) LET $x=1$ IN THE ABOVE EXPANSION

$$\sqrt{1+\tfrac{1}{8}\times 1} \approx 1 + \tfrac{1}{16}\times 1 - \tfrac{1}{512}\times 1^2$$

$$\sqrt{\tfrac{9}{8}} \approx \frac{543}{512}$$

NOW WE HAVE TWO POSSIBLE APPROXIMATIONS

$$\frac{3}{2\sqrt{2}} \approx \frac{543}{512}$$

$\div 3$

$$\frac{1}{2\sqrt{2}} \approx \frac{181}{512}$$

$\times 2$

$$\frac{1}{\sqrt{2}} \approx \frac{181}{256}$$

RATIONALISE LHS.

$$\sqrt{2} \approx \frac{256}{181}$$

$$\frac{\sqrt{2}}{2} \approx \frac{181}{256}$$

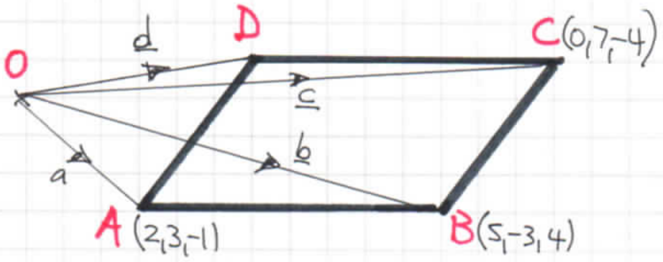
$\times 2$

$$\sqrt{2} \approx \frac{181}{128}$$

IYGB - SYNOPTIC PAPER A - QUESTION 8

a) DRAWING A PARALLELOGRAM

A POSITION VECTOR APPROACH IS AS FOLLOWS



$\Longrightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$\Longrightarrow \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$ (PARALLELOGRAM)

$\Longrightarrow \underline{d} = \underline{a} + (\underline{c} - \underline{b})$

$\Longrightarrow \underline{d} = \underline{a} + \underline{c} - \underline{b}$

$\Longrightarrow \underline{d} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 7 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 13 \\ -9 \end{pmatrix}$

$\therefore \underline{d} = -3\underline{i} + 13\underline{j} - 9\underline{k}$

THERE ARE SEVERAL OTHER APPROACHES (MOST ARE NO MORE THAN INSPECTION)

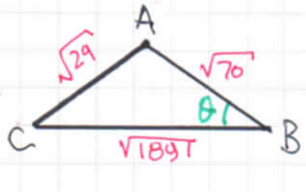
b) • $|\overrightarrow{AC}| = |\underline{c} - \underline{a}| = \left| \begin{pmatrix} 0 \\ 7 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right| = \left| \begin{matrix} -2 \\ 4 \\ -3 \end{matrix} \right| = \sqrt{4+16+9} = \sqrt{29}$

• $|\overrightarrow{AB}| = |\underline{b} - \underline{a}| = \left| \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right| = \left| \begin{matrix} 3 \\ -6 \\ 5 \end{matrix} \right| = \sqrt{9+36+25} = \sqrt{70}$

• $|\overrightarrow{BC}| = |\underline{c} - \underline{b}| = \left| \begin{pmatrix} 0 \\ 7 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} \right| = \left| \begin{matrix} -5 \\ 10 \\ - \end{matrix} \right| = \sqrt{25+100+64} = \sqrt{189}$

BY THE COSINE RULE

$(\sqrt{29})^2 = \sqrt{70}^2 + \sqrt{189}^2 - 2 \times \sqrt{70}\sqrt{189}\cos\theta$



$\cos\theta = \dfrac{70+189-29}{2\sqrt{70}\sqrt{189}}$

$\theta \approx 0.999811\ldots$

$\therefore \hat{ABC} \simeq 1.11°$

IYGB - SYNOPTIC PAPER A - QUESTION 9

a) BY LONG DIVISION OR MANIPULATION

$$\frac{x^2+3}{x-1} = \frac{x(x-1)+(x-1)+4}{x-1} = x+1+\frac{4}{x-1}$$

$A=1$
$B=1$
$C=4$

ALTERNATIVE BY COMPARING COEFFICIENTS

$$\frac{x^2+3}{x-1} \equiv Ax+B+\frac{C}{x-1}$$

$$\frac{x^2+3}{x-1} \equiv \frac{Ax(x-1)+B(x-1)+C}{x-1}$$

$$x^2+3 \equiv Ax^2+(B-A)x+(C-B)$$

$A=1$ $B-A=0$ $C-B=3$
$B-1=0$ $C-1=3$
$B=1$ $C=4$

b) USING PART (a) WE HAVE

$$\int_2^4 \frac{x^2+3}{x-1}\,dx = \int_2^4 x+1+\frac{4}{x-1}\,dx = \left[\frac{1}{2}x^2+x+4\ln|x-1|\right]_2^4$$

$$= (8+4+4\ln 3)-(2+2+\cancel{4\ln 1})$$

$$= 8+4\ln 3$$

IYGB - SYNOPTIC PAPER A - QUESTION 10

START BY REDUCING THE DOUBLE ARGUMENTS

$$\Rightarrow \frac{\cos 2x}{1+\cos 2x} = 1 - 2\tan x$$

- $\cos 2A \equiv \cos^2 A - \sin^2 A$
- $\cos 2A \equiv 2\cos^2 A - 1$
- $\cos 2A \equiv 1 - 2\sin^2 A$

$$\Rightarrow \frac{\cos^2 x - \sin^2 x}{1 + (2\cos^2 x - 1)} = 1 - 2\tan x$$

$$\Rightarrow \frac{\cos^2 x - \sin^2 x}{2\cos^2 x} = 1 - 2\tan x$$

$$\Rightarrow \frac{\cos^2 x - \sin^2 x}{\cos^2 x} = 2 - 4\tan x$$

$$\Rightarrow 1 - \tan^2 x = 2 - 4\tan x$$

$$\Rightarrow 0 = 1 - 4\tan x + \tan^2 x$$

THE QUADRATIC DOES NOT FACTORIZE NICELY, SO PROCEED BY THE QUADRATIC FORMULA OR BY COMPLETING THE SQUARE

$$\Rightarrow (\tan x - 2)^2 - 4 + 1 = 0$$

$$\Rightarrow (\tan x - 2)^2 = 3$$

$$\Rightarrow \tan x - 2 = \pm\sqrt{3}$$

$$\Rightarrow \tan x = \begin{cases} 2+\sqrt{3} \\ 2-\sqrt{3} \end{cases} \Rightarrow \arctan(2\pm\sqrt{3}) = \begin{cases} \frac{5\pi}{12} \\ \frac{\pi}{12} \end{cases}$$

SOLVING SEPARATELY IN RADIANS

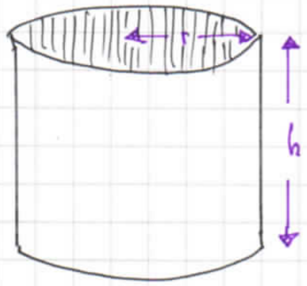
- $x = \frac{5\pi}{12} \pm n\pi$
- $x = \frac{\pi}{12} \pm n\pi$

$n = 0, 1, 2, 3, \ldots$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

IYGB - SYNOPTIC PAPER A - QUESTION 11

a)



$\boxed{\text{CAPACITY} = 1500}$

$\Rightarrow$ VOLUME $= 1500$

$\Rightarrow \pi r^2 h = 1500$

$\Rightarrow \pi r h = \dfrac{1500}{r}$

$$\boxed{A = \pi r^2 + 2\pi r h}$$

BASE CURVED SURFACE

$\Rightarrow A = \pi r^2 + 2\pi r h.$

$\Rightarrow A = \pi r^2 + 2(\pi r h)$

$\Rightarrow A = \pi r^2 + 2\left(\dfrac{1500}{r}\right)$

$\Rightarrow A = \pi r^2 + \dfrac{3000}{r}$ // AS REQUIRED

b) DIFFERENTIATE THE "AREA" EXPRESSION WITH RESPECT TO r

$\Rightarrow A = \pi r^2 + 3000 r^{-1}$

$\Rightarrow \dfrac{dA}{dr} = 2\pi r - 3000 r^{-2}$

FOR STATIONARY VALUES $\dfrac{dA}{dr} = 0$

$\Rightarrow 2\pi r - \dfrac{3000}{r^2} = 0$

$\Rightarrow 2\pi r = \dfrac{3000}{r^2}$

IYGB - SYNOPTIC PAPER A - QUESTION 11

$\Longrightarrow 2\pi r^3 = 3000$

$\Longrightarrow \pi r^3 = 1500$

$\Longrightarrow r = \sqrt[3]{\frac{1500}{\pi}}$

$\Longrightarrow r \simeq 7.82 \text{ cm}$

c) USING THE SECOND DERIVATIVE TEST

$$\frac{dA}{dr} = 2\pi r - 3000r^{-2}$$

$$\frac{d^2A}{dr^2} = 2\pi + 6000r^{-3}$$

$$\left.\frac{d^2A}{dr^2}\right|_{r=7.82} = 2\pi + 6000(7.82\ldots)^{-3} = 6\pi > 0$$

INDEED THE MIN VALUE OF A

d) FINALLY USING $A = \pi r^2 + \frac{3000}{r}$ WITH $r = 7.82$

$$A_{MIN} = \pi(7.82\ldots)^2 + \frac{3000}{(7.82\ldots)}$$

$$A_{MIN} \simeq 576 \text{ cm}^2$$

IYGB – SYNOPTIC PAPER A – QUESTION 12

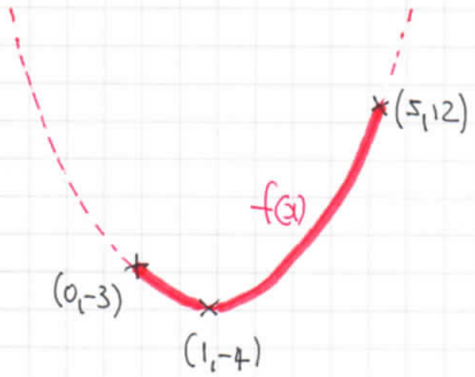
a) LOCATE THE MINIMUM OF THE QUADRATIC BY COMPLETING THE SQUARE

$$f(x) = x^2 - 2x - 3, \quad 0 \leq x \leq 5$$

$$f(x) = (x-1)^2 - 1 - 3$$

$$f(x) = (x-1)^2 - 4$$

SKETCH THE FUNCTION f



∴ RANGE IS :

$$f(x) \in \mathbb{R}, \quad -4 \leq f(x) \leq 12$$

b)

$$f(x) = x^2 - 2x - 3, \quad 0 \leq x \leq 5$$

$$g(x) = ax^2 + 2, \quad x \in \mathbb{R}$$

$$gf(1) = 6$$

$$g(1^2 - 2 \times 1 - 3) = 6$$

$$g(-4) = 6$$

$$a(-4)^2 + 2 = 6$$

$$16a + 2 = 6$$

$$16a = 4$$

$$a = \frac{1}{4}$$

IYGB - SYNOPTIC PAPER A - QUESTION 13

SOLVING THE EQUATIONS SIMULTANEOUSLY

$$\left.\begin{array}{l} y = 2e^{-x} \\ y = e^{x}-1 \end{array}\right\} \Rightarrow e^{x}-1 = 2e^{-x}$$

$$\Rightarrow e^{x}-1 = \frac{2}{e^{x}}$$

$$\Rightarrow A-1 = \frac{2}{A} \qquad (A = e^{x})$$

$$\Rightarrow A^2 - A = 2$$

$$\Rightarrow A^2 - A - 2 = 0$$

$$\Rightarrow (A+1)(A-2) = 0$$

$$\Rightarrow A = \begin{cases} -1 \\ 2 \end{cases}$$

$$\Rightarrow e^{x} = \begin{cases} \cancel{-1} \\ 2 \end{cases}$$

$$\Rightarrow x = \ln 2$$

USING $e^{x}=2$ INTO THE SECOND EQUATION YIELDS $y=1$

$$\therefore P(\ln 2, 1)$$

ALTERNATIVE

- $y = 2e^{-x}$

$$\Rightarrow \frac{y}{2} = e^{-x}$$

$$\Rightarrow \frac{2}{y} = e^{x}$$

- $y = e^{x}-1$

$$\Rightarrow e^{x} = y+1$$

$$\Rightarrow \frac{2}{y} = y+1$$

$$\Rightarrow 2 = y^2 + y$$

IYGB - SYNOPTIC PAPER A - QUESTION 13

$$\Longrightarrow y^2 + y - 2 = 0$$

$$\Longrightarrow (y-1)(y+2) = 0$$

$$\Longrightarrow y = \begin{cases} 1 \\ -2 \end{cases}$$

$$e^x = y + 1$$

$$e^x = \begin{cases} 2 \\ \cancel{-1} \end{cases}$$

$$x = \ln 2$$

HENCE $P(\ln 2, 1)$ AS BEFORE

IYGB - SYNOPTIC PAPER A - QUESTION 14

a) IDENTIFYING THE TRANSFORMATIONS

- TRANLATION BY 1 UNIT TO THE "RIGHT" : $x \longmapsto x-1$

$$4x(x-1) \longmapsto 4(x-1)[(x-1)-1]$$
$$4(x-1)(x-2)$$

- HORIZONTAL STRETCH BY SCALE FACTOR $\frac{2}{3}$: $x \longmapsto \frac{3}{2}x$

$$4(x-1)(x-2) \longmapsto 4\left[\left(\tfrac{3}{2}x\right)-1\right]\left[\left(\tfrac{3}{2}x\right)-2\right]$$
$$4\left(\tfrac{3}{2}x-1\right)\left(\tfrac{3}{2}x-2\right)$$
$$4\left(\tfrac{9}{4}x^2-3x-\tfrac{3}{2}x+2\right)$$
$$9x^2-12x-6x+8$$

$$\therefore\ g(x) = 9x^2-18x+8$$

b) IDENTIFYING AND REVERSING THE TRANSFORMATIONS

- REVERSING A VERTICAL STRETCH BY SCALE FACTOR OF 2

WE HAVE A VERTICAL STRETCH BY SCALE FACTOR $\frac{1}{2}$

$$4x(x-1) \longmapsto \tfrac{1}{2}[4x(x-1)] = 2x(x-1)$$

- REVERSING A TRANSLATION BY 1 UNIT TO THE "RIGHT"

WE NEED A TRANSLATION BY 1 UNIT TO THE "LEFT"

$$2x(x-1) \longmapsto 2(x+1)[(x+1)-1]$$
$$\longmapsto 2x(x+1)$$

$$\therefore\ h(x) = 2x(x+1) = 2x^2+2x$$

IYGB - SYNOPTIC PAPER A - QUESTION 15

a) COMPLETING THE SQUARE IN x & y SO ANSWER CAN BE READ

$$\Rightarrow x^2+y^2-12x-2y+33=0$$
$$\Rightarrow x^2-12x+y^2-2y+33=0$$
$$\Rightarrow (x-6)^2-6^2+(y-1)^2-1^2+33=0$$
$$\Rightarrow (x-6)^2-36+(y-1)^2-1+33=0$$
$$\Rightarrow (x-6)^2+(y-1)^2=4$$

$\therefore$ CENTRE AT $(6,1)$ & RADIUS 2

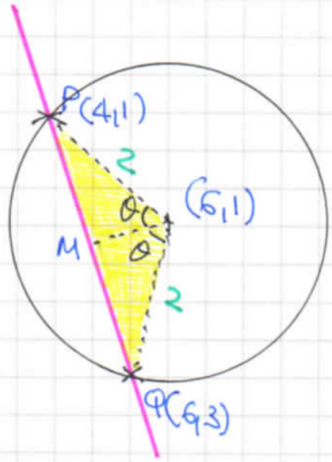
b) SOLVING SIMULTANEOUSLY WE OBTAIN

$$\Rightarrow (x-6)^2+(y-1)^2=4$$
$$\Rightarrow (x-6)^2+(x-3-1)^2=4 \qquad (y=x-3)$$
$$\Rightarrow (x-6)^2+(x-4)^2=4$$
$$\Rightarrow x^2-12x+36+x^2-8x+16=4$$
$$\Rightarrow 2x^2-20x+48=0$$
$$\Rightarrow x^2-10x+24=0$$
$$\Rightarrow (x-6)(x-4)=0$$
$$\Rightarrow x = \begin{cases} 4 \\ 6 \end{cases} \qquad y = \begin{cases} 1 \\ 3 \end{cases}$$

$\therefore$ $P(4,1)$ & $Q(6,3)$

IYGB - SYNOPTIC PAPER A - QUESTION 15

c) START BY DRAWING A DIAGRAM



- DISTANCE PQ

$$|PQ| = \sqrt{(1-3)^2 + (4-6)^2}$$
$$= \sqrt{4+4}$$
$$= \sqrt{8} = 2\sqrt{2}$$

- BY THE COSINE RULE OR SIMPLE TRIGONOMETRY

$$\sin\theta = \frac{|MP|}{2}$$
$$\sin\theta = \frac{\sqrt{2}}{2}$$
$$\theta = 45° = \frac{\pi}{4}$$

- AREA OF SECTOR IS $\frac{1}{2}r^2\theta^c$

$$\frac{1}{2} \times 2^2 \times \underbrace{\frac{\pi}{2}}_{2\theta} = \pi$$

- AREA OF TRIANGLE IN YELLOW

$$\frac{1}{2} \times 2 \times 2 \times \sin(2\theta) = 2\sin\frac{\pi}{2} = 2$$

- REQUIRED AREA = AREA OF SECTOR − AREA OF TRIANGLE

$$= \pi - 2$$

// AS REQUIRED

IYGB - SYNOPTIC PAPER A - QUESTION 16

SOLVING SIMULTANEOUSLY

$$x^2 - \frac{1}{2}y^2 = 1 \qquad y = x + c$$

$$\Rightarrow \qquad \Leftarrow$$

$$x^2 - \frac{1}{2}(x+c)^2 = 1$$

$$\Rightarrow 2x^2 - (x+c)^2 = 2$$

$$\Rightarrow 2x^2 - (x^2 + 2cx + c^2) - 2 = 0$$

$$\Rightarrow 2x^2 - x^2 - 2cx - c^2 - 2 = 0$$

$$\Rightarrow x^2 - 2cx - (c^2+2) = 0$$

NOW USING THE DISCRIMINANT WITH $A=1$, $B=-2c$, $C=-(c^2+2)$

$$\Rightarrow B^2 - 4AC = (-2c)^2 - 4 \times 1 \times [-c^2 - 2]$$

$$= 4c^2 + 4c^2 + 8$$

$$= 8c^2 + 8$$

$$= \geqslant 8$$

AS THE DISCRIMINANT IS POSITIVE (AT LEAST 8) FOR ALL VALUES OF C THE CURVE & LINE INTERSECT REGARDLESS

IYGB - SYNOPTIC PAPER A - QUESTION 17

$$5\frac{dy}{dx} = 2y^2 - 7y + 3$$

START BY SEPARATING VARIABLES

$$\Longrightarrow 5\,dy = (2y^2 - 7y + 3)\,dx$$

$$\Longrightarrow \frac{5}{2y^2 - 7y + 3}\,dy = 1\,dx$$

$$\Longrightarrow \frac{5}{(2y-1)(y-3)}\,dy = 1\,dy$$

PARTIAL FRACTIONS ON THE L.H.S OF THE O.D.E

$$\Longrightarrow \frac{5}{(2y-1)(y-3)} \equiv \frac{P}{2y-1} + \frac{Q}{y-3}$$

$$\Longrightarrow \boxed{5 \equiv P(y-3) + Q(2y-1)}$$

- IF $y = 3 \Longrightarrow 5 = 5Q$
  $\Longrightarrow Q = 1$
- IF $y = 0 \Longrightarrow 5 = -3P - Q$
  $\Longrightarrow 5 = -3P - 1$
  $\Longrightarrow 3P = -6$
  $\Longrightarrow P = -2$

RETURNING TO THE O.D.E

$$\Longrightarrow \int \frac{1}{y-3} - \frac{2}{2y-1}\,dy = \int 1\,dx$$

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$$\Longrightarrow \ln|y-3| - \ln|2y-1| = x + C$$

$$\Longrightarrow \ln\left|\frac{y-3}{2y-1}\right| = x + C$$

$$\Longrightarrow \frac{y-3}{2y-1} = e^{x+C}$$

$$\Longrightarrow \frac{y-3}{2y-1} = Ae^{x} \quad , \quad \text{WHERE } A = e^{C}$$

$$\Longrightarrow y - 3 = 2Aye^{x} - Ae^{x}$$

$$\Longrightarrow Ae^{x} - 3 = 2Aye^{x} - y$$

$$\Longrightarrow Ae^{x} - 3 = y\left[2Ae^{x} - 1\right]$$

$$\Longrightarrow y = \frac{Ae^{x} - 3}{2Ae^{x} - 1}$$

AS REQUIRED

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IF IN ARITHMETIC PROGRESSION, IN THE ORDER GIVEN

$$\Longrightarrow \frac{\pi}{4} - \theta = \phi - \frac{\pi}{4}$$

$$\Longrightarrow \phi + \theta = \frac{\pi}{2}$$

NOW WE HAVE

$$(\sin\theta - \sin\phi)^2 + (\cos\theta + \cos\phi)^2$$

$$= \sin^2\theta - 2\sin\theta\sin\phi + \sin^2\phi + \cos^2\theta + 2\cos\theta\cos\phi + \cos^2\phi$$

$$= (\sin^2\theta + \cos^2\theta) + (\sin^2\phi + \cos^2\phi) + 2[\cos\theta\cos\phi - \sin\theta\sin\phi]$$

$$= 2 + 2[\cos\theta\cos\phi - \sin\theta\sin\phi]$$

$$= 2 + 2[\cos(\theta + \phi)]$$

$$= 2 + \cancel{2\cos\frac{\pi}{2}}$$

$$= 2$$

I.E $k = 2$

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a) I)

$$x = a + \tan t \qquad\qquad y = b + \cot^2 t$$

$$\frac{dx}{dt} = \sec^2 t \qquad\qquad \frac{dy}{dt} = 2\cot t\left(-\operatorname{cosec}^2 t\right)$$

$$\frac{dx}{dt} = \frac{1}{\cos^2 t} \qquad\qquad \frac{dy}{dt} = -\frac{2\cot t}{\sin^2 t}$$

$$\frac{dy}{dt} = -2\frac{\cos t}{\sin t}\cdot\frac{1}{\sin^2 t}$$

$$\frac{dy}{dt} = -\frac{2\cos t}{\sin^3 t}$$

THUS

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-2\cos t}{\sin^3 t}}{\frac{1}{\cos^2 t}} = -\frac{2\cos^3 t}{\sin^3 t} = -2\cot^3 t$$

II)

$$x = a + \tan t \qquad\qquad y = b + \cot^2 t$$

$$x - a = \tan t \qquad\qquad y - b = \cot^2 t$$

$$\tan^2 t = (x-a)^2 \qquad\qquad \tan^2 t = \frac{1}{y-b}$$

$$(x-a)^2 = \frac{1}{y-b}$$

$$(x-a)^2(y-b) = 1$$

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b) STARTING WITH THE LINE WE HAVE

$y = 6x+2$	$y = 6x+2$
$2 = 6x+2$	$5 = 6x+2$
$6x = 0$	$3 = 6x$
$x = 0$	$x = \frac{1}{2}$
$\therefore (0,2)$	$\therefore (\frac{1}{2},5)$

NOW USING THE CARTESIAN EQUATION OF C WITH EACH OF THE ABOVE POINTS

$$\left.\begin{array}{l} (0,2) \Rightarrow (2-b)a^2 = 1 \\ (\frac{1}{2},5) \Rightarrow (5-b)(\frac{1}{2}-a)^2 = 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} 2-b = \frac{1}{a^2} \\ 5-b = \frac{1}{(\frac{1}{2}-a)^2} \end{array}\right\} \text{SUBTRACTING}$$

$$\Rightarrow -3 = \frac{1}{a^2} - \frac{1}{(\frac{1}{2}-a)^2}$$

$$\Rightarrow 3 = \frac{1}{(\frac{1}{2}-a)^2} - \frac{1}{a^2}$$

$$\Rightarrow 3 = \frac{4}{(1-2a)^2} - \frac{1}{a^2}$$

$$\Rightarrow 3(1-2a)^2a^2 = 4a^2 - (1-2a)^2$$

$$\Rightarrow 3a^2(4a^2-4a+1) = 4a^2 - (4a^2-4a+1)$$

$$\Rightarrow 12a^4 - 12a^3 + 3a^2 = 4a - 1$$

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$$\Rightarrow 12a^4 - 12a^3 + 3a^2 - 4a + 1 = 0$$

BY INSPECTION $a=1$ IS A SOLUTION — LONG DIVISION OR ALGEBRAIC MANIPULATION

$$\Rightarrow 12a^3(a-1) + 3a(a-1) - (a-1) = 0$$

$$\Rightarrow (a-1)(12a^3 + 3a - 1)$$

c) $a = 1$ & USING $2 - b = \frac{1}{a^2}$

$$2 - b = 1$$

$$b = 1$$

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START BY OBTAINING THE GRADIENT FUNCTION

$$\Longrightarrow y = x(\ln x)^{\frac{1}{2}}$$

$$\Longrightarrow \frac{dy}{dx} = 1 \times (\ln x)^{\frac{1}{2}} + \cancel{x} \times \frac{1}{2}(\ln x)^{-\frac{1}{2}} \times \frac{1}{\cancel{x}}$$

$$\Longrightarrow \frac{dy}{dx} = (\ln x)^{\frac{1}{2}} + \frac{1}{2}(\ln x)^{-\frac{1}{2}}$$

$$\Longrightarrow \frac{dy}{dx} = \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}}$$

NOW WE ARE GIVEN THAT THE EQUATION OF THE TANGENT, AT THE POINT WHERE $x = a$ IS $4y = bx - a$

THIS LEADS TO TWO EQUATIONS

- $P(a, a\sqrt{\ln a})$ MUST SATISFY THE CURVE AND THE TANGENT
- $\left.\frac{dy}{dx}\right|_{x=a} = \sqrt{\ln a} + \frac{1}{2\sqrt{\ln a}}$
- THE GRADIENT OF THE TANGENT IS $\frac{b}{4}$ (BY INSPECTION)

THUS WE NOW HAVE

$$\underbrace{\sqrt{\ln a} + \frac{1}{2\sqrt{\ln a}}}_{\text{GRADIENT AT } P} = \underbrace{\frac{b}{4}}_{\text{GRADIENT OF TANGENT}}$$

AND

$$4y = bx - a$$

$$4a\sqrt{\ln a} = ba - a$$

$$4\sqrt{\ln a} = b - 1$$

$$\sqrt{\ln a} = \frac{b}{4} - \frac{1}{4}$$

$$\frac{b}{4} = \frac{1}{4} + \sqrt{\ln a}$$

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BY SUBSTITUTION WE GET

$$\Longrightarrow \cancel{\sqrt{\ln a}} + \frac{1}{2\sqrt{\ln a}} = \frac{1}{4} + \cancel{\sqrt{\ln a}}$$

$$\Longrightarrow 2\sqrt{\ln a} = 4$$

$$\Longrightarrow \sqrt{\ln a} = 2$$

$$\Longrightarrow \ln a = 4$$

$$\Longrightarrow a = e^4$$

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$$\int \frac{1-x}{\sqrt{x}(x+1)^2}\,dx = \ldots$$

USING THE GIVEN SUBSTITUTION

$$\boxed{\begin{aligned} \sqrt{x} &= \tan\theta \\ x &= \tan^2\theta \\ dx &= 2\tan\theta\sec^2\theta \end{aligned}}$$

$$\ldots = \int \frac{1-\tan^2\theta}{\cancel{\tan\theta}(\tan^2\theta+1)^2}\left(2\cancel{\tan\theta}\sec^2\theta\,d\theta\right)$$

$$= \int \frac{2\sec^2\theta(1-\tan^2\theta)}{(\sec^2\theta)^2}\,d\theta$$

$$= \int \frac{2\sec^2\theta(1-\tan^2\theta)}{\sec^4\theta}\,d\theta$$

$$= \int \frac{2(1-\tan^2\theta)}{\sec^2\theta}\,d\theta$$

CONVERT THE INTEGRAL INTO $\sec\theta$, SO IT MAY BE SPLIT

$$= \int \frac{2\left[1-(\sec^2\theta-1)\right]}{\sec^2\theta}\,d\theta$$

$$= \int \frac{2(2-\sec^2\theta)}{\sec^2\theta}\,d\theta$$

$$= \int 2\left(\frac{2}{\sec^2\theta}-\frac{\sec^2\theta}{\sec^2\theta}\right)d\theta$$

$$= \int 2\left[2\cos^2\theta-1\right]d\theta \qquad \cos2\theta \equiv 2\cos^2\theta-1$$

$$= \int 2\cos2\theta\,d\theta$$

$$= \sin2\theta + C$$

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$$= 2\sin\theta\cos\theta + C$$

$$= \frac{2\sin\theta}{\cos\theta} \times \cos^2\theta + C$$

$$= 2\tan\theta \times \frac{1}{\sec^2\theta} + C$$

$$= \frac{2\tan\theta}{1+\tan^2\theta} + C$$

$$= \frac{2\sqrt{x}}{1+x} + C$$