IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper B

Difficulty Rating: 3.8150

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 24 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

Prove that the square of a positive integer can never be of the form $3k+2$, $k \in \mathbb{N}$. **(4)**

Question 2

It is given that

$$p = \log_6 25 \qquad \text{and} \qquad q = \log_6 2 .$$

Simplify each of the following logarithms, giving the final answers in terms of p, q and positive integers, where appropriate.

i. $\log_6 (200)$. **(2)**

ii. $\log_6 (3.2)$. **(3)**

iii. $\log_6 (75)$. **(3)**

Question 3

Solve the following quadratic equation.

$$(2x+3)^2 - (4-x)^2 = 45 .$$ **(5)**

Question 4

A cylinder has a radius of $\left(\dfrac{1}{\sqrt{2}-1}\right)$ cm and a height of $\left(\sqrt{2}+1\right)$ cm.

Show, by detailed working, that the volume of this cylinder is exactly

$$\pi\left(7+5\sqrt{2}\right)\text{cm}^3 .$$ **(6)**

Question 5 (****)

The functions f and g are defined as

$$f(x) = 3\left(2^{-x}\right) - 1, \quad x \in \mathbb{R}, \quad x \geq 0$$

$$g(x) = \log_2 x, \quad x \in \mathbb{R}, \quad x \geq 1.$$

a) Sketch the graph of f.

- Mark clearly the exact coordinates of any points where the curve meets the coordinate axes. Give the answers, where appropriate, in exact form in terms of logarithms base 2.
- Mark and label the equation of the asymptote to the curve. **(4)**

b) State the range of f. **(1)**

c) Find $f(g(x))$ in its simplest form. **(3)**

Question 6

In a geometric series the sum of the second and fourth term is 156.

In the same geometric series the sum of the third and the fifth term is 234.

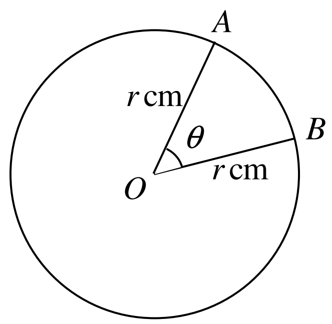
Find the first term and the common ratio of the series. **(7)**

Question 7

By using the substitution $u = 2x^{\frac{3}{2}} - 1$, or otherwise, find an expression for the integral

$$\int \frac{6x^2}{2x^{\frac{3}{2}} - 1} \, dx.$$ **(6)**

Question 8



The figure above shows a circle with centre at O and radius r cm.

The **minor** sector AOB subtends an angle of θ radians at O.

The area of the **minor sector** AOB is 48cm^2.

The length of the **minor arc** AB is 12cm.

Determine the value of r and the value of θ. **(5)**

Question 9

Differentiate $\dfrac{1}{x^2-2x}$ from first principles. **(7)**

Question 10

A curve has implicit equation

$$\frac{3x^2}{y} - 5y = 2(x+8), \quad x \in \mathbb{R},\ y \in \mathbb{R},\ y \neq 0.$$

Find the coordinates of the stationary point of the curve. **(8)**

Question 11

A population p, in millions, is thought to obey the differential equation

$$\frac{dp}{dt} = kp\cos kt$$

where k is a positive constant, and t is measured in days from a certain instant.

When $t = 0$, $p = p_0$.

a) Solve the differential equation to find p in terms of p_0, k and t. **(7)**

The value of k is now assumed to be 3.

b) Calculate, correct to the nearest minute, the time for the population to reach p_0 again, for the first time since $t = 0$. **(6)**

Question 12

A curve is defined by the parametric equations

$$x = \frac{t+3}{t+1}, \quad y = \frac{2}{t+2}, \quad t \in \mathbb{R},\ t \neq -1,\ t \neq -2.$$

Show, with detailed workings, that …

a) … $\dfrac{dy}{dx} = \left(\dfrac{t+1}{t+2}\right)^2$. **(6)**

b) … a Cartesian equation for the curve is given by

$$y = \frac{2(x-1)}{x+1}.$$ **(4)**

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Question 13

The functions f and g are defined by

$$f(x) = 3\ln 2x, \ x \in \mathbb{R}, \ x > 0$$

$$g(x) = 2x^2 + 1, \ x \in \mathbb{R}.$$

Show that the value of the gradient on the curve with equation $y = gf(x)$, at the point where $x = \mathrm{e}$, is given by

$$\frac{36}{\mathrm{e}}(1 + \ln 2).$$ **(6)**

Question 14

It is given that

$$\int_k^{2k} \frac{3x-5}{x(x-1)} \, dx = \ln 72,$$

determine the value of k, $0 < k < 1$. **(10)**

Question 15

The points $A(2,-1,4)$, $B(0,-5,10)$, $C(3,1,3)$ and $D(6,7,-8)$ are referred relative to a fixed origin O.

a) Use vector algebra to show that three of the above four points are collinear. **(4)**

A triangle is drawn using three of the above four points as its vertices.

b) Given further that the triangle has the largest possible area, determine, in exact surd form, the length of its shortest side. **(5)**

Question 16

Use a detailed method to show that

$$\arccos\left(5^{-\frac{1}{2}}\right)+\arccos\left(10^{-\frac{1}{2}}\right)=\frac{3\pi}{4}.$$ **(7)**

Question 17

The rectangle $ABCD$ has three of its vertices located at $A(5,10)$, $B(3,k)$ and $C(9,2)$, where k is a constant.

a) Show that

$$(10-k)(2-k)+12=0.$$

and hence determine the two possible values of k. **(7)**

It is further given that the rectangle $ABCD$ reduces to a square, for one of the two values of k found in part **(a)**.

For the square $ABCD$ ...

b) ... determine its area. **(3)**

c) ... state the coordinates of D. **(2)**

Question 18

The cubic curve with equation

$$y=ax^3+bx^2+cx+d,$$

where a, b, c are non zero constants and d is a constant, has one local maximum and one local minimum.

Show clearly that

$$b^2>3ac$$ **(7)**

Question 19

A circle has equation

$$x^2 + y^2 - 4x - 6y + 8 = 0.$$

The straight line T_1 is a tangent to the circle at the point $P(4,4)$.

a) Find an equation of T_1. **(8)**

The tangent T_1 passes through the point $Q(2,8)$.

The straight line T_2 is a tangent to the circle at the point R and it also passes through the point Q.

b) Determine in any order

i. … the coordinates of R. **(3)**

ii. … an equation of T_2. **(3)**

Question 20

It is given that

$$\sin P + \sin Q \equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right).$$

a) Prove the validity of the above trigonometric identity by using the compound angle identities for $\sin(A+B)$ and $\sin(A-B)$. **(4)**

b) Hence, or otherwise, solve the trigonometric equation

$$\sin 7x + \sin x = 0,\quad 0 \le x < \pi,$$

giving the answers in terms of π. **(5)**

Question 21

In the convergent binomial expansion of

$$(1+bx)^n, \quad |bx|<1$$

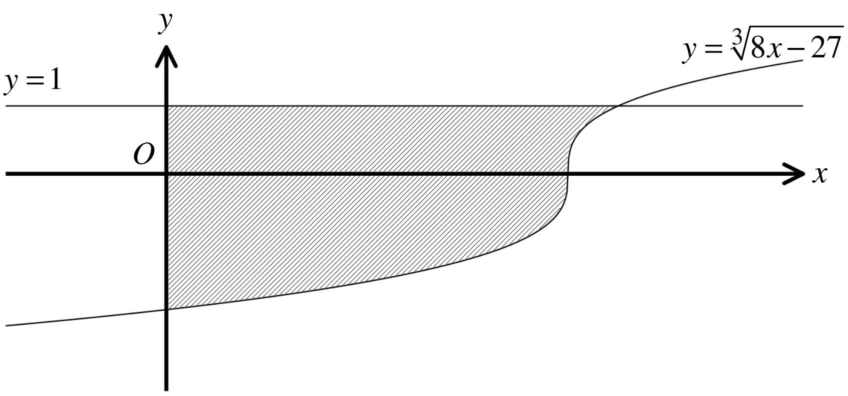
the coefficient of x is -6 and the coefficient of x^2 is 27.

a) Show that $b=3$ and find the value of n. **(7)**

b) Find the coefficient of x^3. **(1)**

c) State the range of values of x for which the above expansion is valid. **(1)**

Question 22



The figure above shows the curve C with equation

$$y=\sqrt[3]{8x-27}, \quad x \in \mathbb{R}.$$

The finite region R, shown shaded in the figure, is bounded by C, the y axis and the straight line with equation $y=1$.

When the lengths are measured in m, R models the design of a yacht rudder.

Show that the area of the yacht rudder is 11 m^2. **(10)**

Question 23

A curve has equation

$$y = \arcsin 2x\,,\ -\frac{1}{2} \le x \le \frac{1}{2}\,,\ -\frac{\pi}{2} \le y \le \frac{\pi}{2}\,.$$

a) By finding $\frac{dx}{dy}$ and using an appropriate trigonometric identity show that

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}\,. \qquad \textbf{(4)}$$

b) Show further that …

i. … $\frac{d^2y}{dx^2} = \frac{Ax}{\left(1-4x^2\right)^{\frac{3}{2}}}$, **(2)**

ii. … $\frac{d^3y}{dx^3} = \frac{Bx^2+C}{\left(1-4x^2\right)^{\frac{5}{2}}}$,

where A, B and C are constants to be found. **(4)**

Question 24

Find in exact form the solution of the following equation.

$$e^2 - e^{3x} - 1 = \left(\frac{e^{x+1}}{e^{-2x}}\right)^2,\ x \in \mathbb{R}\,. \qquad \textbf{(10)}$$
