IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper C

Difficulty Rating: 3.8925

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

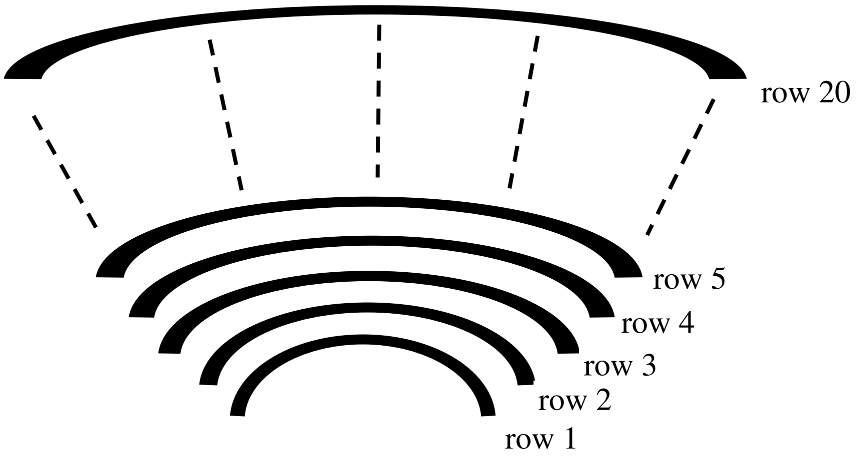
The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 22 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

Seats in a theatre are arranged in rows. The number of seats in this theatre form the terms of an arithmetic series.



The sixth row has 23 seats and the fifteenth row has 50 seats.

a) Find the number of seats in the first row. **(4)**

The theatre has 20 rows of seats in total.

b) Find the number of seats in this theatre. **(2)**

Question 2

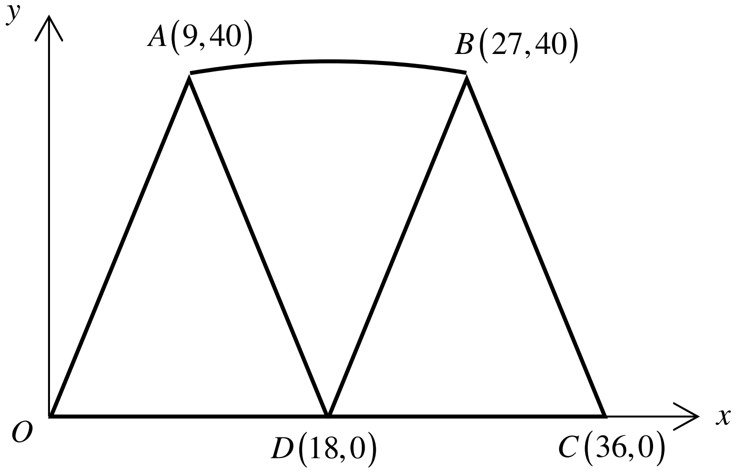
a) Use integration by parts to find

$$\int x\cos\left(\tfrac{1}{2}x\right)\,dx.$$ **(3)**

b) Hence determine

$$\int x^2\sin\left(\tfrac{1}{2}x\right)\,dx.$$ **(3)**

Question 3



The figure above shows the cross section of a river dam modelled in a system of coordinate axes where all units are in metres.

The cross section of the dam consists of a circular sector ADB and two isosceles triangles OAD and DBC.

The coordinates of the points A, B, C and D are $(9,40)$, $(27,40)$, $(36,0)$ and $(18,0)$, respectively.

a) Find the length of AD. **(2)**

b) Show that the angle ADB is approximately 0.4426 radians. **(2)**

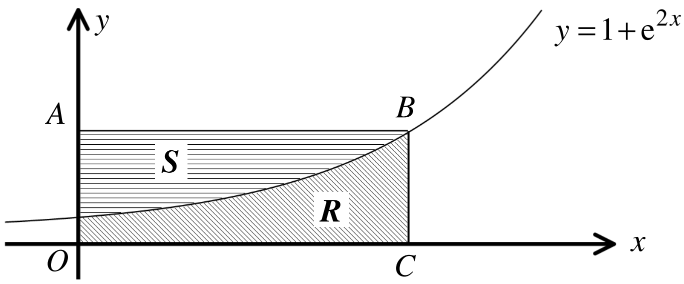
c) Hence determine, to the nearest m^2, the cross sectional area of the dam. **(4)**

Question 4

Use the substitution $t = \sqrt{1-x^3}$ to show that

$$\int x^5 \sqrt{1-x^3}\, dx = -\frac{2}{45}\left(3x^3+2\right)\left(1-x^3\right)^{\frac{3}{2}} + C.$$ **(10)**

Question 5



The figure above shows the graph of the curve with equation

$$y = 1 + e^{2x}, \ x \in \mathbb{R}.$$

The point C has coordinates $(1,0)$. The point B lies on the curve so that BC is parallel to the y axis. The point A lies on the y axis so that $OABC$ is a rectangle.

The region R is bounded by the curve, the coordinate axes and the line BC.

The region S is bounded by the curve, the y axis and the line AB.

Show that the area of R is equal to the area of S. **(7)**

Question 6

The first term of a geometric progression is 1200 and its sum to infinity is 1600.

a) Find the sum of the first five terms of the progression. **(4)**

The n^{th} term of the progression is denoted by u_n.

b) Evaluate the sum

$$\sum_{r=6}^{\infty} u_r \,.$$ **(2)**

Question 7

The curve C_1 has equation

$$y = |x-1|, \ x \in \mathbb{R}.$$

The curve C_2 has equation

$$y = |2x+1|, \ x \in \mathbb{R}.$$

a) Sketch the graph of C_1 and the graph of C_2 in the same set of axes, indicating the coordinates of any intercepts of the graphs with the coordinate axes. **(4)**

b) Hence, solve the inequality

$$|2x+1| \geq |x-1|.$$ **(4)**

Question 8

A curve has equation

$$y = \frac{1}{4}e^{2x-3} - 4\ln\left(\frac{1}{2}x\right), \quad x > 0.$$

The tangent to the curve, at the point where $x = 2$, crosses the coordinate axes at the points A and B.

Show that the area of the triangle OAB, where O is the origin, is given by

$$\frac{(16-3e)^2}{16(4-e)}.$$ **(10)**

Question 9

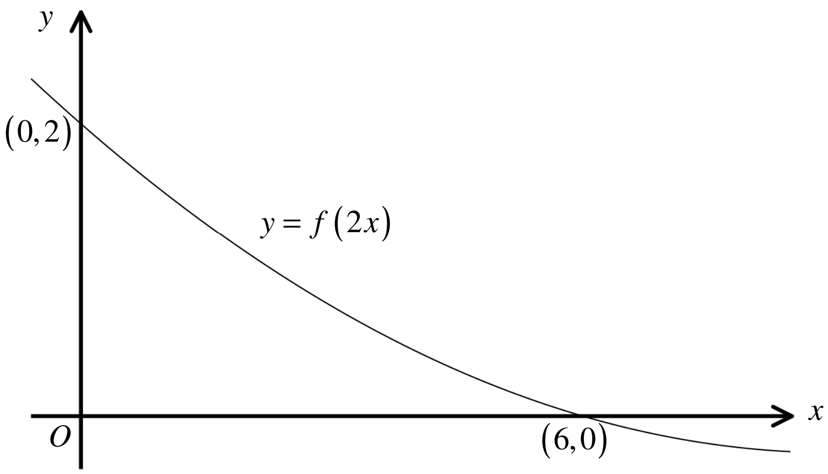
At the point P, which lies on the curve with equation

$$y^3 - y^2 = e^x,$$

the gradient is $\frac{6}{5}$.

Determine the possible coordinates of P. **(10)**

Question 10



The figure above shows part of the curve with equation $y = f(2x)$.

The curve meets the coordinate axes at $(6,0)$ and $(0,4)$.

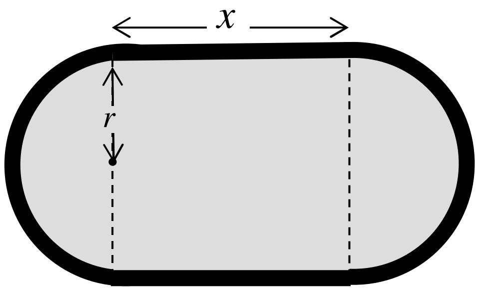
a) Sketch the graph of $y = f(x)$.

The sketch must include the coordinates of any points where the graph meets the coordinate axes. **(4)**

b) Sketch on separate diagram the graph of $y = f(4x-1)$.

The sketch must include the coordinates of the point where the graph meets the x axis. **(2)**

Question 11



The figure above shows the design of an athletics track inside a stadium.

The track consists of two semicircles, each of radius r m, joined up to a rectangular section of length x metres.

The total length of the track is 400 m and encloses an area of A m^2.

a) By obtaining and manipulating expressions for the total length of the track and the area enclosed by the track, show that

$$A = 400r - \pi r^2 .$$ **(4)**

In order to hold field events safely, it is required for the area enclosed by the track to be as large as possible.

b) Determine by **differentiation** an exact value of r for which A is stationary. **(4)**

c) Show that the value of r found in part **(b)** gives the maximum value for A. **(2)**

d) Show further that the maximum area the area enclosed by the track is

$$\frac{40000}{\pi} \text{ m}^2 .$$ **(2)**

The calculations for maximizing the area of the field within the track are shown to a mathematician. The mathematician agrees that the calculations are correct but he feels the resulting shape of the track might not be suitable.

e) Explain, by calculations, the mathematician's reasoning. **(2)**

Question 12

A circle C_1 has equation

$$x^2 + y^2 + 20x - 2y + 52 = 0.$$

a) Determine the coordinates of the centre of C_1 and the size of its radius. **(4)**

A different circle C_2 has its centre at $(14,8)$ and the size of its radius is 10.

The point P lies on C_1 and the point Q lies on C_2.

b) Determine the minimum distance of PQ. **(3)**

Question 13

A cubic curve C_1 has equation

$$y = (x-7)\left(x^2 + 2x - 3\right).$$

A quadratic curve C_2 has equation

$$y = (2x+5)(7-x).$$

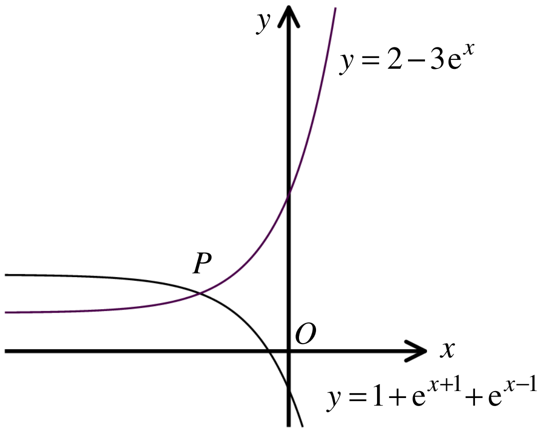
a) Sketch on separate set of axes the graphs of C_1 and C_2.

The sketches must contain the coordinates of the points where each of the curves meet the coordinate axes. **(4)**

b) Hence, find in exact form where appropriate, the three solutions of the following equation.

$$(x-7)\left(x^2 + 2x - 3\right) = (2x+5)(7-x).$$ **(7)**

Question 14



The figure above shows the graphs of

$$y = 2 - 3e^x \qquad \text{and} \qquad y = 1 + e^{x+1} + e^{x-1}.$$

The graphs meet at the point P.

Show that the y coordinate of P is

$$\frac{2e^2 + 3e + 2}{e^2 + 3e + 1}.$$

(6)

Question 15

Use trigonometric algebra to solve the equation

$$\sin\left[\arcsin\frac{1}{4} + \arccos x\right] = 1.$$

You may not use any calculating aid in this question. **(4)**

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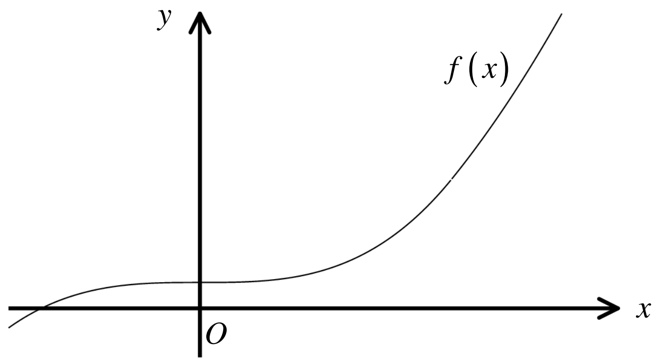
Question 16



The figure above shows the graph of a function $f(x)$, defined by

$$f(x) = \begin{cases} ax^3 + 2, & x \in \mathbb{R},\ x \le 2 \\ bx^2 - 2, & x \in \mathbb{R},\ x > 2 \end{cases}$$

The function is **continuous** and **smooth.**

Find the value of a and the value of b. **(7)**

Question 17

Solve the following simultaneous logarithmic equations.

$$3\log_8(xy) = 4\log_2 x$$

$$\log_2 y = 1 + \log_2 x$$ **(8)**

Question 18

Consider the following identity for t.

$$\frac{1}{t\left(t^2+1\right)} \equiv \frac{At+B}{t^2+1}+\frac{C}{t}.$$

a) Find the value of each of the constants A, B and C. **(4)**

In a chemical reaction the mass, m grams, of the chemical produced at time t, in minutes, satisfies the differential equation

$$\frac{dm}{dt}=\frac{m}{t\left(t^2+1\right)}.$$

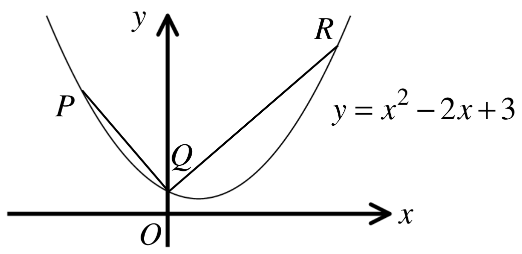
b) Find a general solution of the differential equation, in the form $m=f(t)$. **(6)**

Two minutes after the reaction started the mass produced is 10 grams.

c) Calculate the mass which will be produced after a further period of 2 minutes. **(4)**

d) Determine, in exact surd form, the maximum mass that will ever be produced by this chemical reaction. **(2)**

Question 19



The figure above shows the curve C with equation

$$y = x^2 - 2x + 3.$$

The points $P(-1,6)$, $Q(0,3)$ and R all lie on C.

Given that $\measuredangle PQR = 90°$, determine the exact coordinates of R. **(9)**

Question 20

A curve has equation

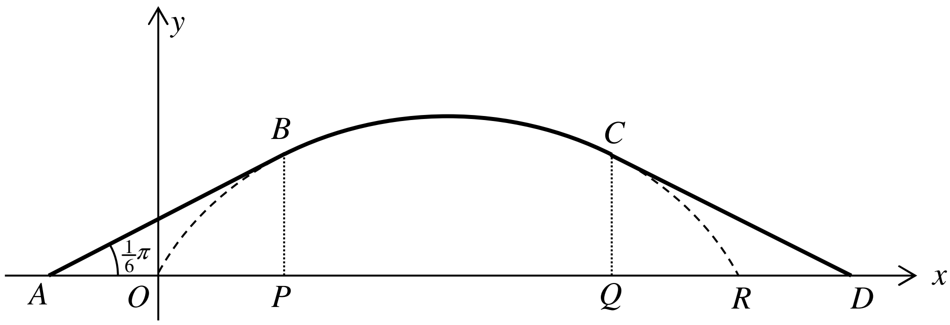
$$f(x) \equiv 4^{ax+b}, \ x \in \mathbb{R},$$

where a and b are non zero constants.

Find the value of a and the value of b, given further that

$$f\left(\tfrac{2}{3}\right) = \tfrac{1}{4}\sqrt[3]{4} \quad \text{and} \quad f\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{2}.$$ **(8)**

Question 21



The figure above shows a **symmetrical** design for a suspension bridge arch $ABCD$.

The curve $OBCR$ is a cycloid with parametric equations

$$x = 6(2t - \sin 2t), \quad y = 6(1 - \cos 2t), \quad 0 \le t \le \pi .$$

a) Show clearly that

$$\frac{dy}{dx} = \cot t \ . \qquad (5)$$

b) Find the in exact form the length of OR. **(2)**

c) Determine the maximum height of the arch. **(2)**

The arch design consists of the curved part BC and the straight lines AB and CD.

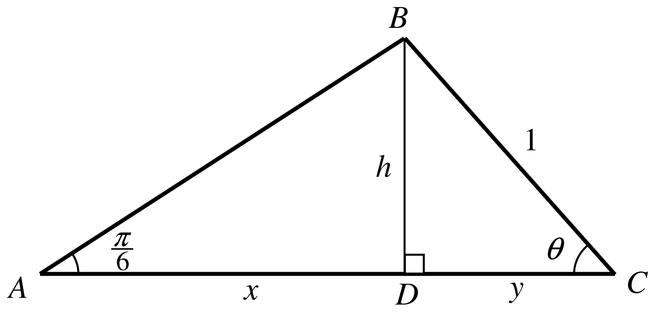
The straight lines AB and CD are tangents to the cycloid at the points B and C.

The angle BAO is $\frac{1}{6}\pi$.

d) Find the value of t at B, by considering the gradient at that point. **(3)**

e) Find, in exact form, the length of the straight line AD. **(6)**

Question 22



The figure above shows a triangle ABC, where $\measuredangle BAC = \frac{1}{6}\pi$, $\measuredangle BCA = \theta$ and $|BC| = 1$. The straight line segment BD, labelled as h, is perpendicular to AC.

Let $AD = x$ and $DC = y$.

a) By expressing h in terms of θ, and x in terms of h, show that

$$x + y = \sqrt{3}\sin\theta + \cos\theta,$$

and hence deduce that the area of the triangle ABC is given by

$$\sin\theta\sin\left(\theta + \tfrac{1}{6}\pi\right).$$ **(7)**

b) By using the trigonometric identities for

$$\cos\left[\theta + \left(\theta + \tfrac{1}{6}\pi\right)\right] \quad \text{and} \quad \cos\left[\theta - \left(\theta + \tfrac{1}{6}\pi\right)\right],$$

write a simplified expression for the area of the triangle ABC. **(5)**

The value of θ can vary.

c) By using part **(b)**, deduce that the maximum value of the area of the triangle ABC is

$$\tfrac{1}{4}\left(2 + \sqrt{3}\right)$$

and this maximum value occurs when $\theta = \frac{5}{12}\pi$. **(3)**