IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper E

Difficulty Rating: 3.895

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 21 questions in this question paper.
The total mark for this paper is 200.

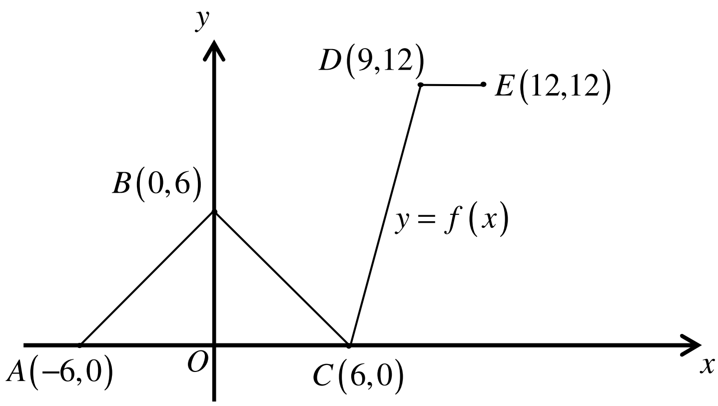
Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

The figure below shows the graph of a function with equation $y = f(x)$.

The graph consists of four straight line segments joining the points $A(-6,0)$, $B(0,6)$, $C(6,0)$, $D(9,12)$ and $E(12,12)$.



Sketch on separate diagrams the graph of …

a) … $y = f(x-6)$. **(2)**

b) … $y = f(-x)$. **(2)**

c) … $y = \frac{1}{2} f\left(\frac{1}{2}x\right)$. **(2)**

Each sketch must include the new coordinates of A, B, C, D and E.

Question 2

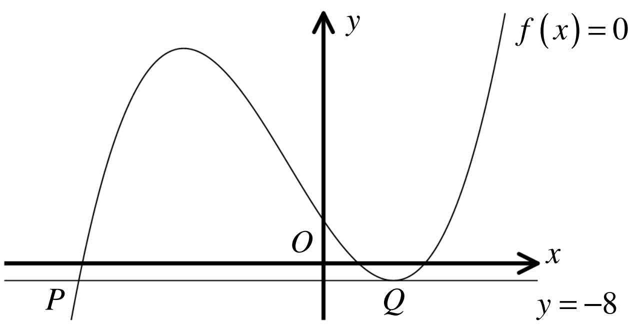
$$f(x) \equiv x^3 + 3x^2 - 24x + 20, \; x \in \mathbb{R}.$$

a) Show that $(x-1)$ is a factor of $f(x)$. **(2)**

b) Hence factorize $f(x)$ as the product of a linear and a quadratic factor. **(2)**

c) Find, in exact form where appropriate, the solutions of the equation $f(x) = 0$. **(3)**

The straight line with equation $y = -8$ **touches** the graph of $f(x)$ at the point $Q(2,-8)$ and crosses the graph of $f(x)$ at the point P, as shown in the figure below.



d) Determine the coordinates of P. **(5)**

Question 3

$$x = \ln(\sec 3y), \; 0 < y < \tfrac{1}{6}\pi.$$

Determine, with full justification, an expression for $\frac{dy}{dx}$, in terms of x. **(7)**

Question 4

The table below shows experimental data connecting two variables t and W.

t	1	3	4	7	8	10
W	2.0	4.0	6.5	19.0	34.0	65.0

It is assumed that t and W are related by an equation of the form

$$W = ab^t,$$

where a and b are non zero constants.

a) Find an equation of a straight line, in terms of well defined constants, in order to investigate the validity of this assumption. **(3)**

b) Plot a suitable graph to show that the assumption of part **(a)** is valid. **(2)**

c) Use the information from the graph to estimate, correct to 2 decimal places, the value of a and the value of b. **(4)**

d) Estimate the value of W when $t = 20$. **(1)**

Question 5

The points A and C are the diagonally opposite vertices of a square $ABCD$.

The straight line l_1 with equation

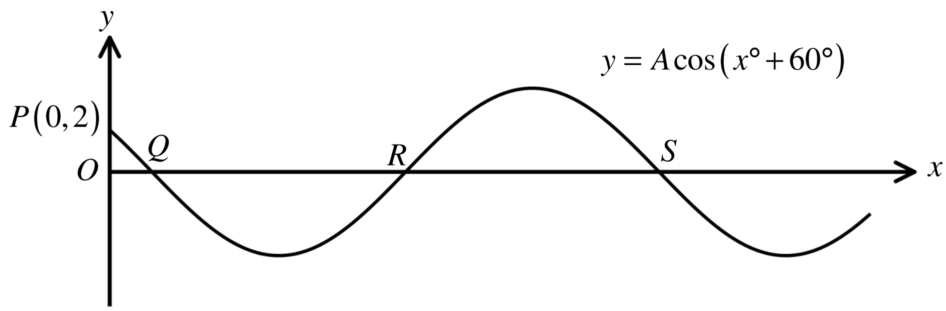
$$3x - 2y = 24.$$

meets the x and y axes at A and C, respectively.

The straight line l_2 passes through B and D.

Determine an equation of l_2. **(8)**

Question 6



The figure above shows part of the graph of the curve with equation

$$y = A\cos\left(x° + 60°\right),$$

where x is measured in degrees and A is a constant.

The point $P(0,2)$ lies on the curve.

a) Find the value of A. **(2)**

The first three x intercepts of the curve, for which $x > 0$, are the points labelled as Q, R and S.

b) State the coordinates of Q, R and S. **(3)**

Question 7

A curve has equation

$$y = e^{-2x} + axe^{-2x},$$

where a is a non zero constant.

Show that the value of $\dfrac{d^2y}{dx^2}$ at the stationary point of the curve is $-2ae^{\frac{2}{a}-1}$. **(8)**

Question 8

It is given that

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B.$$

a) Use the above trigonometric identity to show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x. \qquad \textbf{(5)}$$

b) Hence find

$$\int \cos x\left(6\sin x - 2\sin 3x\right)^{\frac{2}{3}}\,dx. \qquad \textbf{(5)}$$

Question 9

The straight line with equation

$$y = 2x + c$$

is a tangent to the curve with equation

$$y = x^2 + 6x + 7.$$

By **using the discriminant** of a suitable quadratic, determine the value of the constant c and find the point of contact between the tangent and the curve. **(8)**

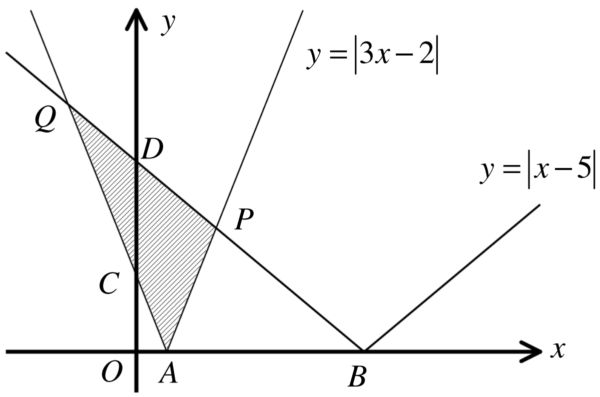
Question 10

$$\cot A = -\frac{3}{4} \quad \text{and} \quad \cos B = \frac{5}{13}.$$

If A is reflex and B is also reflex, show that

$$\tan(A+B) = \frac{56}{33}. \qquad (7)$$

Question 11



The figure above shows the graphs of

$$C_1: \; y = |3x-2|, \; x \in \mathbb{R},$$

$$C_2: \; y = |x-5|, \; x \in \mathbb{R}.$$

a) State the coordinates of the points where each of the graphs meet …

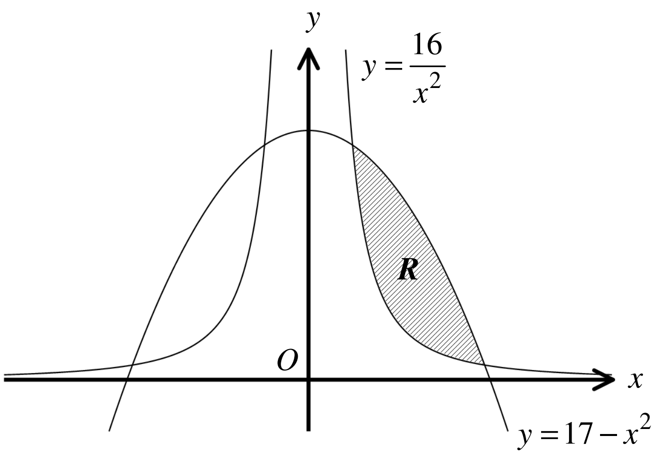
i. … the x axis, indicted by A and B. **(2)**

ii. … the y axis, indicted by C and D. **(2)**

The two graphs intersect at the points P and Q.

b) Find the exact area of the triangle APQ. **(6)**

Question 12



The figure above shows the graphs of the curves with equations

$$y=\frac{16}{x^2} \qquad \text{and} \qquad y=17-x^2.$$

The finite region R, shown shaded in the figure above, is bounded by the two curves in the first quadrant.

Find the area of R. **(10)**

Question 13

A rectangular piece of card has length x cm and an area of 1200 cm^2.

A square of side length 5 cm is removed from each corner and the sides of the remaining card are folded upwards to form an **open** box of height 5 cm.

The resulting box must have a volume greater than 2850 cm^3.

a) Show clearly that

$$x^2-73x+1200<0.$$ **(4)**

b) Hence determine the range of the possible values of x. **(3)**

Question 14

$$\frac{dy}{dx} = \frac{y^2 - 1}{x}, \quad x > 0, \quad y > 0$$

Show that the solution of the above differential equation subject to $y = 2$ at $x = 1$, is

$$y = \frac{3 + x^2}{3 - x^2}.$$ **(10)**

Question 15

A geometric has positive terms and positive common ratio r.

The difference between the first and the fourth term of a geometric progression is five times as large as the difference between its second and its third term.

a) Show that the common ratio r of the progression is a solution of the equation

$$r^3 - 5r^2 + 5r - 1 = 0.$$ **(3)**

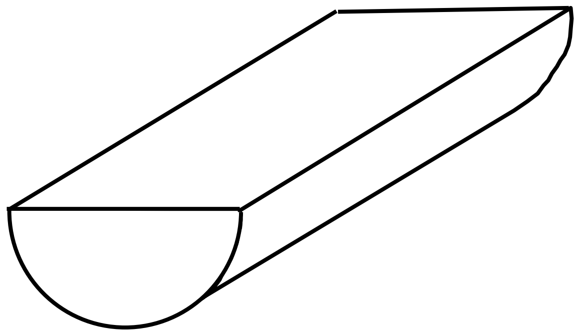
b) Find, in exact surd form where appropriate, the solutions of the above equation. **(5)**

The sum to infinity of the progression is $\sqrt{6} + \sqrt{2}$.

c) Determine, in exact surd form, the first term of the progression. **(4)**

[*detailed working must be shown in this question in order to obtain full credit*]

Question 16



The figure above shows a solid prism, which is in the shape a right semi-circular cylinder.

The total surface area of the 4 faces of the prism is $\sqrt[3]{27\pi}$.

Given that the measurements of the prism are such so that its volume is maximized, find in exact simplified form the volume of the prism. **(12)**

Question 17

$$x = \frac{1}{2}(-1 + 4\tan\theta), \ -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi \ .$$

a) Use trigonometric identities to show that

$$4x^2 + 4x + 17 = 16\sec^2\theta . \qquad \textbf{(4)}$$

b) Hence find the exact value of

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4x^2 + 4x + 17} \, dx . \qquad \textbf{(8)}$$

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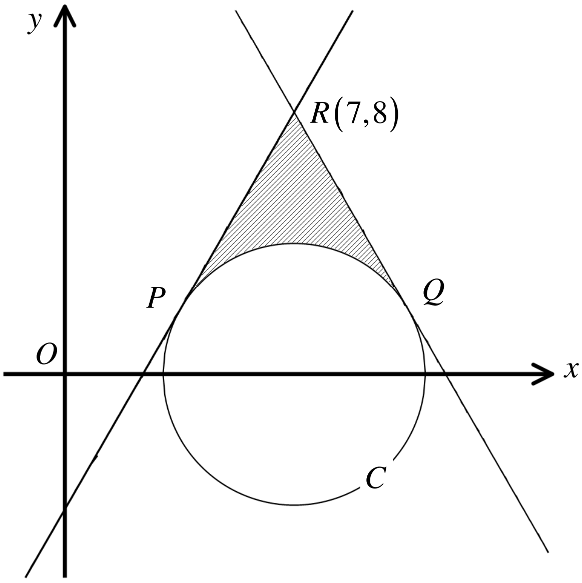
Question 18



The figure above shows the circle C with equation

$$x^2 + y^2 - 14x + 33 = 0.$$

a) Determine the coordinates of the centre of C and the size of its radius. **(3)**

The tangents to C from the point $R(7,8)$ meet C at the points P and Q.

b) Show that the area of the finite region bounded by C and the two tangents, shown shaded in figure, is

$$\frac{16}{3}\left[3\sqrt{3} - \pi\right].$$ **(8)**

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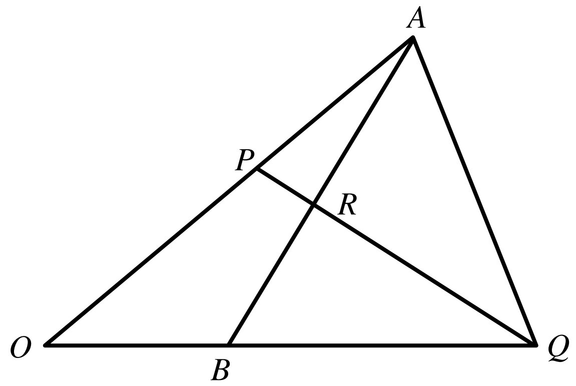
Question 19

The figure below shows a triangle OAQ.



- The point P lies on OA so that $OP : OA = 3 : 5$.
- The point B lies on OQ so that $OB : BQ = 1 : 2$.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

a) Given that $\overrightarrow{AR} = h\overrightarrow{AB}$, where h is a scalar parameter with $0 < h < 1$, show that

$$\overrightarrow{OR} = (1-h)\mathbf{a} + h\mathbf{b}.$$ **(3)**

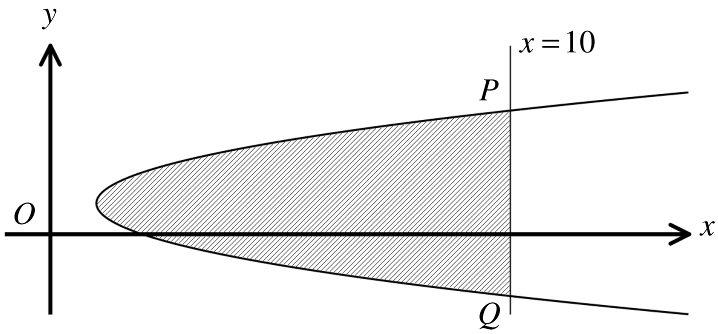
b) Given further that $\overrightarrow{PR} = k\overrightarrow{PQ}$, where k is a scalar parameter with $0 < k < 1$, find a similar expression for $\overrightarrow{OR}$ in terms of k, $\mathbf{a}$, $\mathbf{b}$. **(3)**

c) Determine …

i. … the value of k and the value of h. **(4)**

ii. … the ratio of $\overrightarrow{PR} : \overrightarrow{PQ}$. **(1)**

Question 20



The figure above shows the curve with parametric equations

$$x = t^2 + 1\ ,\quad y = 2t + 2\ ,\quad t \in \mathbb{R}\ .$$

The straight line with equation $x = 10$ meets the curve at the points P and Q.

The area of the finite region bounded by the curve and the straight line with equation $x = 10$ is shown shaded in the figure above.

Show that this area is given by

$$8\int_0^3 t^2 \ dt\ ,$$

and hence find its value. **(10)**

Question 21

Solve each of the following logarithmic equations, giving the answers in exact simplified form where appropriate.

a) $\log_2(256x^2) = 1 + 2\log_2\left(\frac{1}{2}x^4\right)$. (7)

b) $2\log_2\left(\frac{y}{2}\right) + \log_2\sqrt{y} = 8$. (7)

[***detailed working must be shown in this question in order to obtain full credit***]