IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper H

Difficulty Rating: 3.9225/0.6739

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 22 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

The points A, B and C have coordinates $(-1,-4)$, $(0,3)$ and $(14,1)$, respectively.

a) Find an equation of the straight line which passes through B and C, giving the answer in the form $ax+by=c$, where a, b and c are integers. **(3)**

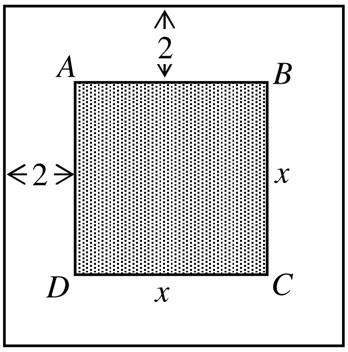
b) Show that AB is perpendicular to BC. **(2)**

A circle is passing through the points A, B and C.

c) Determine the coordinates of the centre of this circle. **(2)**

d) Show that the radius of this circle is $k\sqrt{10}$, where k is a rational number. **(2)**

Question 2



The figure above shows a square piece of lawn $ABCD$ of side length x metres.

The lawn is surrounded by a path which is 2 metres wide, as shown in the figure.

The area of the lawn must be less than the area of the path.

a) Show clearly that

$$x^2-8x-16<0.$$ **(4)**

a) Hence determine the range of the possible values of x, in terms of surds where appropriate. **(6)**

Question 3

$$f(x) \equiv 2x^2 - 4x + 5, \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are integers. **(3)**

b) State the maximum value of $\dfrac{6}{f(x)}$. **(1)**

c) Solve the equation $f(x) = 13$, giving the answers as exact simplified surds. **(3)**

Question 4

The curve C has equation

$$2\cos 3x \sin y = 1, \quad 0 \le x, y \le \pi.$$

The point $P\left(\dfrac{\pi}{12}, \dfrac{\pi}{4}\right)$ lies on C.

Show that an equation of the tangent to C at P is

$$y = 3x.$$

(6)

Question 5

Consider the following sequence

3, 8, 15, 24, 35, 48, …

Prove algebraically, that the product of any 2 consecutive terms of the above sequence can be written as the product of 4 consecutive integers. **(6)**

Question 6

The curve C has equation

$$y = \frac{2}{2 - \sin x}.$$

Show clearly that

$$\frac{dy}{dx} = \frac{1}{2} y^2 \cos x \ . \qquad \textbf{(7)}$$

Question 7

It is given that

$$(\cos x + \sec x)^2 \equiv \cos^2 x + \tan^2 x + 3.$$

a) Prove the validity of the above trigonometric identity. **(3)**

b) Hence solve the trigonometric equation

$$\cos^2 x + \tan^2 x = \frac{13}{4}, \quad 0 \le x < 2\pi,$$

giving the answers in terms of π. **(8)**

Question 8

$$f(x) \equiv \frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 - 1}}, \ x \in \mathbb{R}, \ |x| \ge 1.$$

Find without the use of a calculator the value of

$$f\left(\tfrac{5}{12}\sqrt{6}\right).$$

Detailed workings must be shown in this question. **(6)**

Question 9

$$f(x) = \frac{1}{x}, \; x \in \mathbb{R}, \; x \neq 0.$$

$$g(x) = \frac{1}{x-1} + 1, \; x \in \mathbb{R}, \; x \neq 1.$$

a) Describe mathematically the two transformations that map the graph of $f(x)$ onto the graph of $g(x)$. **(2)**

b) Sketch the graph of $g(x)$.

The sketch must include …

- … the coordinates of any points where $g(x)$ meet the coordinate axes.
- … the equations of any asymptotes of $g(x)$. **(4)**

c) Solve the equation

$$g(x) = x - 1,$$

giving the answers in the form $a + b\sqrt{5}$, where a and b are constants. **(6)**

Question 10

The value V, in £, of a computer system t years after it was purchased is modelled by the following expression

$$V = 100 + A\mathrm{e}^{-kt}, \; t \geq 0,$$

where A and k are positive constants.

Its value after one year was £650 and after a further period of four years £350.

Find, correct to the nearest £, the value of the system when new. **(7)**

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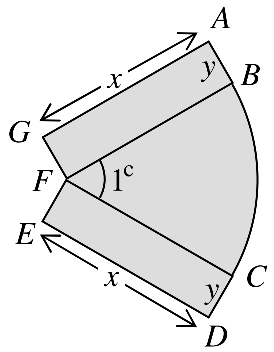
Question 11



The figure above shows a clothes design consisting of two identical rectangles attached to each of the straight sides of a circular sector of radius x cm.

The rectangles measure x cm by y cm and the circular sector subtends an angle of one radian at the centre.

The perimeter of the design is 40 cm.

a) Show that the area of the design, A cm^2, is given by

$$A = 20x - x^2 .$$ **(4)**

b) Determine **by differentiation** the value of x for which A is stationary. **(5)**

c) Show that the value of x found in part **(b)** gives the maximum value for A. **(2)**

d) Find the maximum area of the design. **(1)**

Question 12

$$f(x) = \sqrt{\frac{4-x}{4+x}}, \ |x| < 4.$$

a) Expand $f(x)$ as an infinite convergent series, up and including the term in x^2. **(7)**

b) By substituting $x = 0.5$ in the expansion of part **(a)**, show that

$$\sqrt{7} \approx \frac{339}{128}.$$ **(2)**

Question 13

The curve C is given parametrically by

$$x = 1 - 3t, \quad y = \frac{t+6}{t+2}, \quad t \in \mathbb{R}.$$ **(5)**

a) Find a simplified expression for $\frac{dy}{dx}$, in terms of t.

b) Show that the straight line L with equation

$$4x - 3y = 1$$

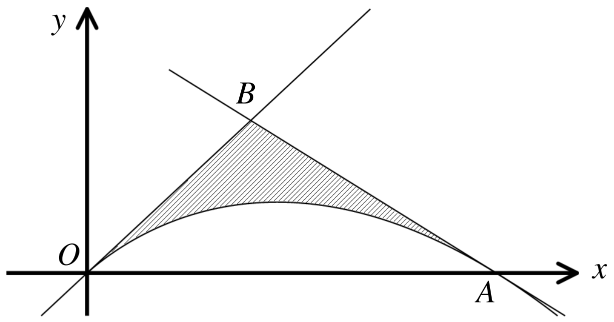
is a tangent to C, and determine the coordinates of the point of tangency between L and C. **(7)**

Question 14

Solve the following trigonometric equation

$$\arctan\left(\frac{1}{x}\right) + \arctan\left(\frac{1}{x+1}\right) = \frac{\pi}{4}.$$ **(8)**

Question 15



The figure above show the graph of the curve C with equation

$$y = x\left(1 - x^{\frac{2}{3}}\right),\ x \in \mathbb{R},\ x \geq 0.$$

The curve meets the coordinate axes at the origin and at the point $A(1,0)$.

The two tangents to C at the origin O and at the point A, meet at the point B.

a) Calculate the value of $\frac{dy}{dx}$ at O, and hence write down an equation of the tangent to C at O. **(3)**

b) Show that an equation of the tangent to C at A is

$$2x + 3y = 2$$ **(5)**

c) Determine the area of the finite region bounded by C and the tangents to C at O and at A. **(7)**

Question 16

The curve C has equation

$$y = 5x + \frac{4}{x} - 3, \ x \neq 0$$

Show that the straight line with equation

$$y = 4x + 1$$

is a tangent to C, and find the coordinates of the point of tangency. **(7)**

Question 17

$$\frac{dy}{dx} \cot x = 1 - y^2 .$$

Solve the differential equation above, subject to the boundary condition $y = 0$ at $x = \frac{\pi}{4}$, to show that

$$y = \frac{1 - 2\cos^2 x}{1 + 2\cos^2 x} .$$ **(12)**

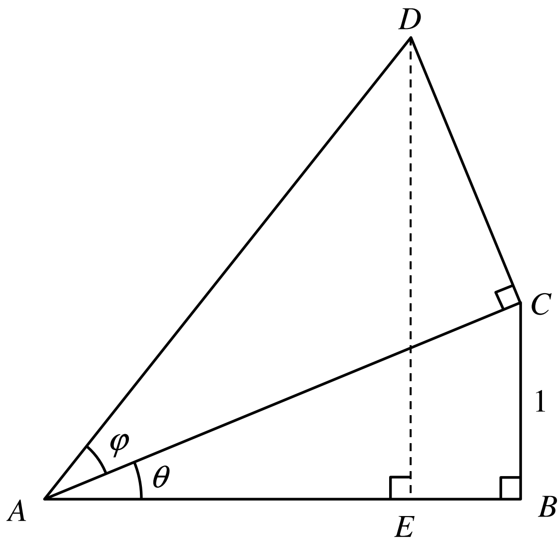
Question 18

The common difference of an arithmetic series is denoted by d and the sum of its first n terms is denoted by S_n.

Show clearly that

$$d = S_{n+2} - 2S_{n+1} + S_n .$$ **(4)**

Question 19



The figure above shows two right angles triangles ABC and ACD. The angles CAB and DAC are denoted by θ and φ, respectively.

The length of BC is 1.

The point E lies on AB so that the angle AED is $90°$.

Show clearly that the length of AE is $\cot\theta - \tan\varphi$. **(7)**

Question 20

$$\int \frac{1}{\sqrt{x^2+x^n}}\,dx\,,\ n \neq 2,\ x \geq 0.$$

a) Show that the substitution $u^2 = 1 + x^{n-2}$ transforms above integral into

$$\frac{1}{n-2}\int \frac{2}{(u-1)(u+1)}\,du\,.$$ **(5)**

b) Use partial fractions to find, in terms of x and n, an integrated expression for the original integral. **(7)**

Question 21

The function f is defined by

$$f(x) = \begin{cases} -x^2 + 8x - 5, & x \in \mathbb{R}, \ x \le 2 \\ x^2 - 2x + 8, & x \in \mathbb{R}, \ x > 2 \end{cases}$$

a) Show that f …

i. … is not continuous.

ii. … is an increasing function. **(6)**

Let the set A be defined

$$A = \{x \in \mathbb{R} : 1 \le x \le 3\}.$$

b) Determine the range of $f(A)$. **(3)**

c) Find an expression for $f^{-1}(x)$, indicating clearly its domain. **(7)**

Question 22

$$\mathbf{a} = \left(\tfrac{1}{2}x^2 + y^2 + 3\right)\mathbf{i} + 4\mathbf{j} \quad \text{and} \quad \mathbf{b} = (x + y)\mathbf{i} + 2\mathbf{j}.$$

Determine the value of x and the value of y given that $\mathbf{a}$ and $\mathbf{b}$ are parallel. **(5)**