IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper I

Difficulty Rating: 3.9625/0.6871

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 23 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

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Question 1

The variables x and y are thought to obey a law of the form

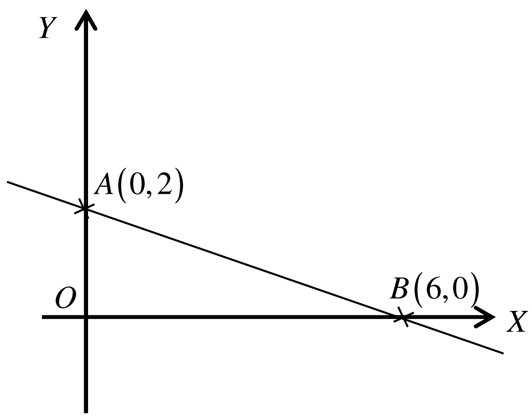
$$y = ax^n,$$

where a and n are non zero constants.

Let $X = \log_{10} x$ and $Y = \log_{10} y$.

a) Show there is a linear relationship between X and Y. **(4)**

The figure below shows the graph of Y against X.



b) Determine the value of a and the value of n. **(3)**

Question 2

A curve C is defined implicitly by

$$(x+y)^3 = 27x, \quad x, y \in \mathbb{R}.$$

Verify that the point on C where $x = 1$ is a stationary point. **(7)**

Question 3

The straight line l_1 has equation $2x + y - 18 = 0$ and crosses the x axis at the point P.

The straight line l_2 is parallel to l_1 and passes through the point $Q(-4,6)$.

The point R is the x intercept of l_2.

a) Determine the coordinates of P and R. **(5)**

The point S lies on l_1 so that RS is perpendicular to l_1.

b) Calculate the area of the triangle PRS. **(7)**

Question 4

Solve the differential equation

$$\frac{dy}{dx} = 4xy - 3yx^2$$

subject to the condition $y = 1$ at $x = 2$, giving the answer in the form $y = f(x)$. **(7)**

Question 5

$$f(x) \equiv x^2 - 4\sqrt{3}x - 15, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $f(x) = (x+a)^2 + b$, where a and b are constants. **(3)**

b) Hence find the exact solutions of the equation $f(x) = 0$. **(4)**

Question 6

A liquid is cooling down and its temperature θ °C satisfies

$$\theta = 20 + 30e^{-\frac{t}{20}}, \ t \geq 0$$

where t is the time in minutes after a given instant.

Find the value of t when the temperature of the liquid has reduced to half its initial temperature. **(7)**

Question 7

Given that k and A are constants with $k > 0$, then

$$(2 - kx)^8 \equiv 256 + Ax + 1008x^2 + \ldots$$

Find the value of k and the value of A. **(7)**

Question 8

The points A and B have coordinates $(-1, 2)$ and $(1, 8)$, respectively.

a) Show that the equation of the perpendicular bisector of AB is

$$3y + x = 15.$$ **(5)**

The points A and B lie on a circle whose centre is at $C(3, k)$.

b) Determine an equation for the circle. **(5)**

Question 9

Simplify, showing clearly all the workings, the trigonometric expression

$$\cos^3\theta\sin\theta - \sin^3\theta\cos\theta,$$

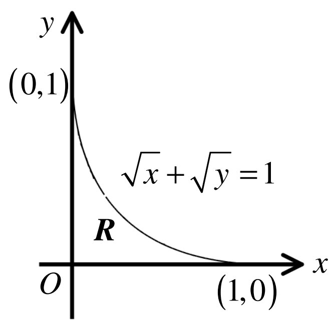
giving the final answer in the form $A\sin k\theta$, where A and k are constants. **(6)**

Question 10

Relative to a fixed origin, the coordinates of three points $A(1,1,1)$, $B(4,-1,3)$ and $C(2,5,-1)$, are given.

Find the position vector of the point P if $4\overrightarrow{PA} + 3\overrightarrow{PB} = 5\overrightarrow{PC}$. **(6)**

Question 11



The figure above shows the curve with equation

$$\sqrt{x}+\sqrt{y}=1\ ,\ x\in\mathbb{R},\ 0\le x\le 1.$$

The curve meets the coordinate axes at the points $(1,0)$ and $(0,1)$.

The finite region R is bounded by the curve and the coordinate axes.

Show that the area of R is $\frac{1}{6}$ **(8)**

Question 12

Given that

$$y = 3\cos(\ln x) + 2\sin(\ln x), \quad x > 0,$$

show clearly that

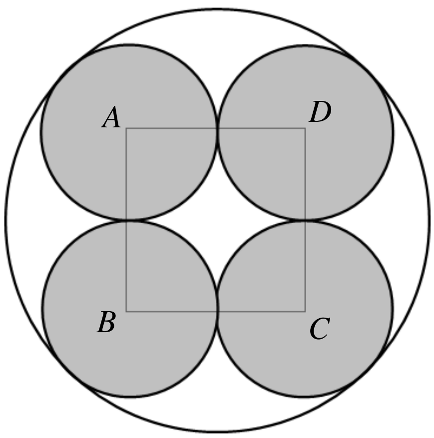
$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} = Ay,$$

stating the value of the constant A. **(10)**

Question 13

Four circles are touching in such a way so that their centres form the corners of a square $ABCD$. These four circles are circumscribed by a larger circle.

This is shown in the figure below.



Show, by detailed workings, that the ratio of the total area of the four smaller circles to the area of the larger circle is given by

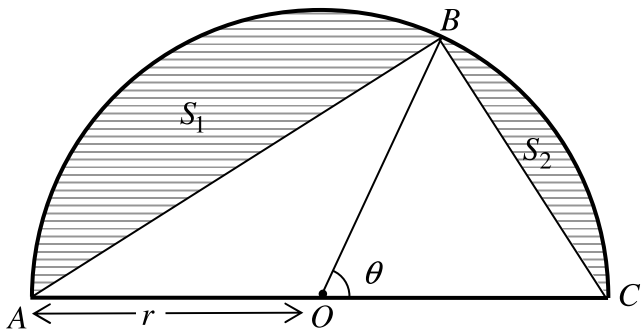
$$12 - 8\sqrt{2} : 1$$ **(10)**

Question 14

$$y = \frac{1}{3\sqrt{x}}\left[\frac{2}{x} - 3\right], \; x > 0.$$

Find the range of values of x for which y is decreasing. **(8)**

Question 15



The figure above shows a semicircle of radius r cm, where AOC is a diameter with point O the centre of the semicircle.

The point B lies on the circular part of the semicircle so that the angle BOC is θ radians.

The chords AB and BC define two segments S_1 and S_2, respectively.

Given that the area of S_1 is four times as large as the area of S_2, show that

$$\pi + 3\sin\theta = 5\theta.$$ **(7)**

Question 16

$$f(x) = \frac{x-2}{x+2}, \; x \in \mathbb{R}.$$

Use the formal definition of the derivative as a limit, to show that

$$f'(x) = \frac{4}{(x+2)^2}.$$ **(7)**

Question 17

Prove that if 1 is added to the product of any 4 consecutive positive integers, the resulting number will always be a square number. **(7)**

Question 18

A solid right circular cone has radius x cm and perpendicular height $6x$ cm.

The cone is heated so that the area of its circular base is expanding at the constant rate of $0.25 \text{ cm}^2 \text{s}^{-1}$.

Find the rate at which the volume of the cone is increasing, when the radius of the base of the cone has reached 2.5 cm.

(*You may assume that the bolt is expanding uniformly when heated*)

$\left[\text{volume of a cone of radius } r \text{ and height } h \text{ is given by } \frac{1}{3}\pi r^2 h\right]$ **(6)**

Question 19

Solve the following exponential equation, giving the answers correct to 3 s.f., where appropriate.

$$3^{t+1} = 6 + 3^{2t-1}$$ **(8)**

Question 20

The function f is defined as

$$f(x) = \ln\left(\frac{1+x}{1-x}\right), \ |x| < 1.$$

a) Show that $f(x)$ is an odd function. **(3)**

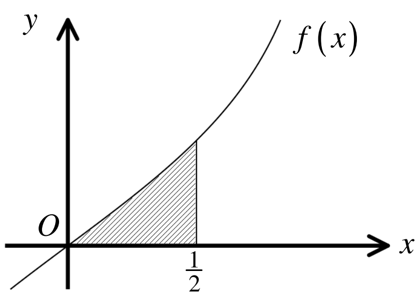
b) Find an expression for $f'(x)$ as a single simplified fraction, showing further that $f'(x)$ is an even function. **(6)**

c) Determine an expression for $f^{-1}(x)$. **(5)**

d) Use the substitution $u = e^x + 1$ to find the exact value of

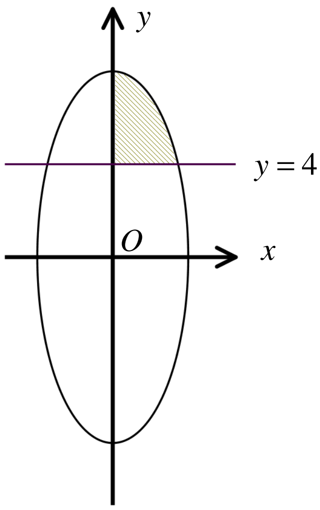
$$\int_0^{\ln 3} f^{-1}(x)\ dx.$$ **(8)**

The figure below shows part of the graph of $f(x)$.



e) Find an exact value for the area of the shaded region, bounded by $f(x)$, the coordinate axes and the straight line with equation $x = \frac{1}{2}$. **(3)**

Question 21



A curve is defined in terms of a parameter θ, by the following equations.

$$x = \cos\theta, \ y = 8\sin\theta, \ 0 \le \theta < 2\pi.$$

Determine an exact value for the area of the finite region bounded by the curve, the y axis for which $y \ge 0$, and the straight line with equation $y = 4$ for which $y \ge 4$. **(10)**

Question 22

It is given that

$$S_n = \sum_{k=1}^{n}\left[\sum_{r=1}^{k}\left(2^r\right)\right].$$

Show that

$$S_n = 2^{n+2} - 2n - 4. \qquad \textbf{(9)}$$

Question 23

By changing the base of the logarithmic integrand into base e and further using integration by parts, show that

$$\int_1^{e} \log_{10} x \, dx = \frac{1}{\ln 10}. \qquad (9)$$

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