IYGB - SYNOPTIC PAPER I - QUESTION 1

a) "TAKING LOGS", BASE 10, FOR THE GIVEN EQUATION

$$\Rightarrow y = ax^n$$
$$\Rightarrow \log_{10} y = \log_{10}(ax^n)$$
$$\Rightarrow \log_{10} y = \log_{10} a + \log_{10} x^n$$
$$\Rightarrow \log_{10} y = \log_{10} a + n\log_{10} x$$
$$\Rightarrow \log_{10} y = n(\log_{10} x) + (\log_{10} a)$$
$$Y = m \cdot X + C$$

$\therefore$ A LINEAR RELATIONSHIP INDEED

b) LOOKING AT THE y INTERCEPT, $A(0,2)$

$$\Rightarrow \log_{10} a = 2$$
$$\Rightarrow a = 10^2$$
$$\Rightarrow a = 100$$

LOOKING AT THE GRADIENT

$$\Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = n$$
$$\Rightarrow \frac{0-2}{6-0} = n$$
$$\Rightarrow n = -\frac{1}{3}$$

IYGB – SYNOPTIC PAPER I – QUESTION 2

DIFFERENTIATE THE EXPRESSION W.R.T x

$$\Longrightarrow (x+y)^3 = 27x$$

$$\Longrightarrow 3(x+y)^2 \times \left(1+\frac{dy}{dx}\right) = 27$$

$$\Longrightarrow \left(1+\frac{dy}{dx}\right)(x+y)^2 = 9$$

FIND THE VALUE OF y WHEN $x=1$

$$(1+y)^3 = 27$$
$$1+y = 3$$
$$y = 2$$

THUS IF $x=1$, $y=2$ THEN $\frac{dy}{dx}=0$ IN $\left(1+\frac{dy}{dx}\right)(x+y)^2=9$

$$\Longrightarrow (1+0)\times(1+2)^2 = 1\times 3^2$$
$$= 9$$
$$= \text{RHS}$$

INDEED STATIONARY //

IYGB – SYNOPTIC PAPER I – QUESTION 3

a) $l_1: 2x+y-18=0 \implies y = -2x+18$

$2x+0-18=0$

$2x=18$

$x=9$

$\therefore P(9,0)$

l_2 HAS THE SAME GRADIENT & PASSES THROUGH $Q(-4,6)$

$\implies y-y_0 = m(x-x_0)$

$\implies y-6 = -2(x+4)$

$\implies y-6 = -2x-8$

$\implies y = -2x-2$

WHEN $y=0$

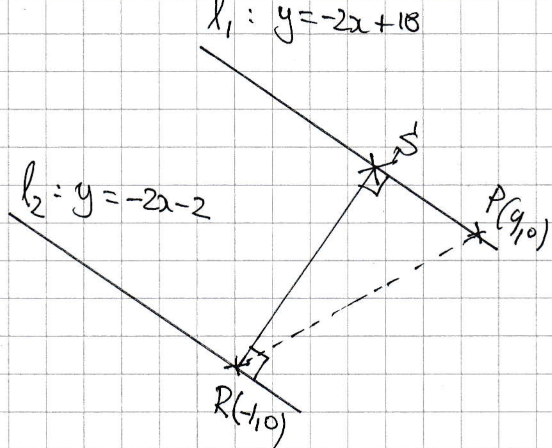
$0=-2x-2$

$2x=-2$

$x=-1$

$\therefore R(-1,0)$

b) DRAWING A DIAGRAM FIRST



- THE GRADIENT OF RS IS $+\frac{1}{2}$
- EQUATION THROUGH $R(-1,0)$ & S IS

$y-y_0 = m(x-x_0)$

$y-0 = \frac{1}{2}(x+1)$

$2y = x+1$

- SOLVING SIMULTANEOUSLY WITH l_1

$\left.\begin{array}{l} y=-2x+18 \\ 2y=x+1 \end{array}\right\} \implies 2(-2x+18) = x+1$

$\implies -4x+36 = x+1$

$\implies 35 = 5x$

$\implies x=7$ & $y=4$ $\quad \therefore S(7,4)$

- LENGTHS OF RS & SP USING $d=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$|RS| = \sqrt{(4-0)^2+(7+1)^2} = \sqrt{16+64} = \sqrt{80} = 4\sqrt{5}$

$|SP| = \sqrt{(4-0)^2+(7-9)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

$\therefore$ TRIANGLE AREA $= \frac{1}{2}(4\sqrt{5})\times(2\sqrt{5}) = 20$

IYGB - SYNOPTIC PAPER I - QUESTION 4

FACTORIZE THE R.H.S & SEPARATE VARIABLES

$$\Rightarrow \frac{dy}{dx} = 4xy - 3yx^2$$

$$\Rightarrow \frac{dy}{dx} = xy(4-3x)$$

$$\Rightarrow \frac{1}{y}\,dy = x(4-3x)\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int 4x - 3x^2\,dx$$

$$\Rightarrow \ln|y| = 2x^2 - x^3 + C$$

$$\Rightarrow y = e^{2x^2 - x^3 + C}$$

$$\Rightarrow y = e^{2x^2 - x^3} \times e^{C}$$

$$\Rightarrow \boxed{y = Ae^{2x^2 - x^3}}$$

APPLY THE BOUNDARY CONDITION (1,2)

$$\Rightarrow 1 = Ae^{8-8}$$

$$\Rightarrow 1 = Ae^{0}$$

$$\Rightarrow A = 1$$

$$\therefore \; y = e^{2x^2 - x^3}$$

IYGB - SYNOPTIC PAPER I - QUESTION 5

a) COMPLETING THE SQUARE

$$f(x) = x^2 - 4\sqrt{3}x - 15 = (x - 2\sqrt{3})^2 - (2\sqrt{3})^2 - 15$$
$$= (x - 2\sqrt{3})^2 - (4 \times 3) - 15$$
$$= (x - 2\sqrt{3})^2 - 27$$

b) SOLVING THE EQUATION USING PART (a)

$$\Rightarrow f(x) = 0$$
$$\Rightarrow (x - 2\sqrt{3})^2 - 27 = 0$$
$$\Rightarrow (x - 2\sqrt{3})^2 = 27$$
$$\Rightarrow x - 2\sqrt{3} = \begin{cases} \sqrt{27} \\ -\sqrt{27} \end{cases}$$
$$\Rightarrow x - 2\sqrt{3} = \begin{cases} 3\sqrt{3} \\ -3\sqrt{3} \end{cases}$$
$$\Rightarrow x = \begin{cases} 5\sqrt{3} \\ -\sqrt{3} \end{cases}$$

IYGB - SYNOPTIC PAPER I - QUESTION 6

$$\theta = 20 + 30e^{-\frac{1}{20}t}$$

WHEN $t=0$ (INITIALLY)

$$\theta = 20 + 30e^{0}$$
$$\theta = 20 + 30 \times 1$$
$$\theta = 50$$

INITIAL TEMPERATURE IS 50°C IE HALF THE INITIAL TEMPERATURE IS 25°C

$$\Rightarrow \quad 25 = 20 + 30e^{-\frac{1}{20}t}$$
$$\Rightarrow \quad 5 = 30e^{-\frac{1}{20}t}$$
$$\Rightarrow \quad \frac{1}{6} = e^{-\frac{1}{20}t}$$
$$\Rightarrow \quad 6 = e^{\frac{1}{20}t}$$
$$\Rightarrow \quad \ln 6 = \frac{1}{20}t$$
$$\Rightarrow \quad t = 20\ln 6$$
$$\simeq 35.8$$

IYGB – SYNOPTIC PAPER I – QUESTION 7

EXPAND AND COMPARE COEFFICIENTS

$$(2-kx)^8 = \binom{8}{0}(2)^8(-kx)^0 + \binom{8}{1}(2)^7(-kx)^1 + \binom{8}{2}(2)^6(-kx)^2 + \dots$$

$$= (1\times 256\times 1) + [8\times 128\times(-kx)] + (28\times 64\times k^2x^2)$$

$$= 256 \underbrace{-1024kx}_{A} + \underbrace{1792k^2x^2}_{1008}$$

EXTRACTING TWO EQUATIONS

$A = -1024k$ & $1792k^2 = 1008$

$$k^2 = \frac{9}{16}$$

$$k = +\frac{3}{4} \quad (k>0)$$

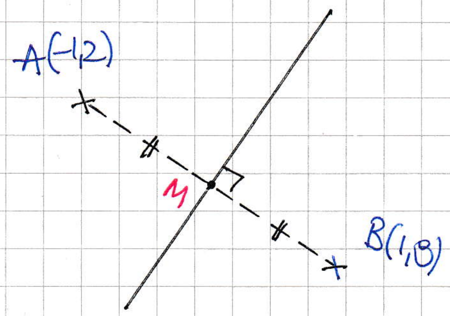
& $A = -1024\times\frac{3}{4}$

$A = -768$

$\therefore$ $A = -768$ & $k = \frac{3}{4}$

IYGB - SYNOPTIC PAPER I - QUESTION 8

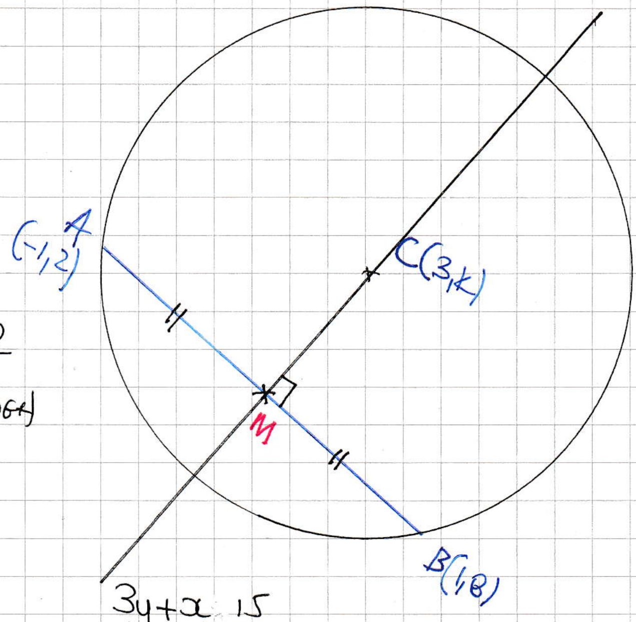
a) LOOKING AT THE DIAGRAM



- MIDPOINT $M\left(\frac{-1+1}{2}, \frac{2+8}{2}\right) = M(0,5)$
- GRADIENT AB $= \frac{8-2}{1-(-1)} = \frac{6}{2} = 3$
- PERPENDICULAR GRADIENT $= -\frac{1}{3}$
- $y - y_0 = m(x - x_0)$

$$y - 5 = -\frac{1}{3}(x - 0)$$
$$y - 5 = -\frac{1}{3}x$$
$$3y - 15 = -x$$
$$3y + x = 15$$

b) LOOKING AT THE 2ND DIAGRAM



- BY CIRCLE THEOREMS, THE PERPENDICULAR BISECTOR OF ANY CHORD MUST PASS THROUGH THE CENTRE

C(3,k) MUST BALANCE

$$3y + x = 15$$
$$3k + 3 = 15$$
$$3k = 12$$
$$k = 4$$

- RADIUS = |AC| OR |BC|

$$|BC| = \sqrt{(k-8)^2 + (3-1)^2} = \sqrt{(4-8)^2 + (3-1)^2} = \sqrt{16+4} = \sqrt{20}$$

- FINALLY WE OBTAIN

$$(x-3)^2 + (y-4)^2 = (\sqrt{20})^2$$
$$(x-3)^2 + (y-4)^2 = 20$$

IYGB - SYNOPTIC PAPER I - QUESTION 9

PROCEED AS FOLLOWS

$$\Longrightarrow \cos^3\theta\sin\theta - \sin^3\theta\cos\theta$$

$$= \cos\theta\sin\theta\left[\cos^2\theta - \sin^2\theta\right]$$

$$= \cos\theta\sin\theta \times \cos 2\theta \qquad \left[\cos 2\theta \equiv \cos^2\theta - \sin^2\theta\right]$$

$$= \frac{1}{2}\left(2\cos\theta\sin\theta\right)\cos 2\theta$$

$$= \frac{1}{2}\sin 2\theta\cos 2\theta \qquad \left[\sin 2\theta \equiv 2\sin\theta\cos\theta\right]$$

$$= \frac{1}{4} \times 2\sin 2\theta\cos 2\theta$$

$$= \frac{1}{4}\sin 4\theta$$

$\sin 4\theta = \sin(2\times 2\theta)$

$\sin 4\theta = 2\sin 2\theta\cos 2\theta$

$A = \frac{1}{4}$, $k = 4$

IYGB - SYNOPTIC PAPER I - QUESTION 10

LET THE POSITION VECTORS BE

$\underline{a} = \underline{i} + \underline{j} + \underline{k}$, $\underline{b} = 4\underline{i} - \underline{j} + 3\underline{k}$, $\underline{c} = 2\underline{i} + 5\underline{j} - \underline{k}$, $\underline{p} = \lambda\underline{i} + \mu\underline{j} + \nu\underline{k}$

NOW WE HAVE

$$4\overrightarrow{PA} + 3\overrightarrow{PB} = 5\overrightarrow{PC}$$

$$\Rightarrow 4(\underline{a} - \underline{p}) + 3(\underline{b} - \underline{p}) = 5(\underline{c} - \underline{p})$$

$$\Rightarrow 4\underline{a} + 3\underline{b} - 4\underline{p} - 3\underline{p} = 5\underline{c} - 5\underline{p}$$

$$\Rightarrow 4\underline{a} + 3\underline{b} - 5\underline{c} = 3\underline{p}$$

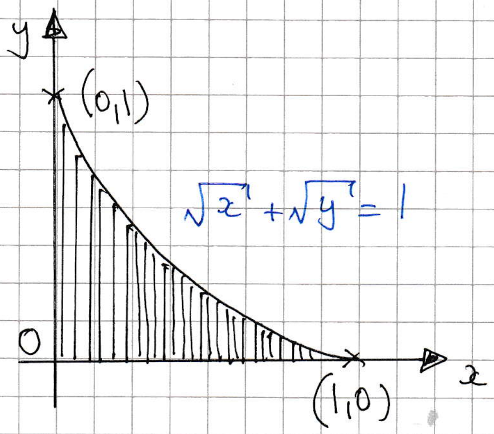
$$\Rightarrow 4(\underline{i} + \underline{j} + \underline{k}) + 3(4\underline{i} - \underline{j} + 3\underline{k}) - 5(2\underline{i} + 5\underline{j} - \underline{k}) = 3\underline{p}$$

$$\Rightarrow 6\underline{i} - 24\underline{j} + 18\underline{j} = 3\underline{p}$$

$$\Rightarrow \underline{p} = 2\underline{i} - 8\underline{j} + 6\underline{k}$$

IYGB - SYNOPTIC PAPER I - QUESTION 11

- WRITE THE EQUATION IN THE FORM $y = f(x)$



$$\sqrt{x} + \sqrt{y} = 1$$
$$\sqrt{y} = 1 - \sqrt{x}$$
$$y = (1 - \sqrt{x})^2$$
$$y = 1 - 2\sqrt{x} + x$$

- $$\text{AREA} = \int_0^1 1 - 2\sqrt{x} + x \, dx$$

$$= \int_0^1 1 - 2x^{\frac{1}{2}} + x \, dx$$

$$= \left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 \right]_0^1$$

$$= \left(1 - \frac{4}{3} + \frac{1}{2}\right) - (0)$$

$$= \frac{1}{6}$$

AS REQUIRED

IYGB - SYNOPTIC PAPER I - QUESTION 12

DIFFERENTIATING WITH RESPECT TO x

$$\frac{dy}{dx} = -3\sin(\ln x) \times \frac{1}{x} + 2\cos(\ln x) \times \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x}\left[2\cos(\ln x) - 3\sin(\ln x)\right]$$

DIFFERENTIATING ONCE MORE

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\left[2\cos(\ln x) - 3\sin(\ln x)\right] + \frac{1}{x}\left[-2\sin(\ln x) \times \frac{1}{x} - 3\cos(\ln x) \times \frac{1}{x}\right]$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\left[2\cos(\ln x) - 3\sin(\ln x)\right] - \frac{1}{x^2}\left[2\sin(\ln x) + 3\cos(\ln x)\right]$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\left[2\cos(\ln x) - 3\sin(\ln x) + 2\sin(\ln x) + 3\cos(\ln x)\right]$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\left[5\cos(\ln x) - \sin(\ln x)\right]$$

FINALLY WE HAVE

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^2 \times \frac{-1}{x^2}\left[5\cos(\ln x) - \sin(\ln x)\right] + x \times \frac{1}{x}\left[2\cos(\ln x) - 3\sin(\ln x)\right]$$

$$= -\left[5\cos(\ln x) - \sin(\ln x)\right] + \left[2\cos(\ln x) - 3\sin(\ln x)\right]$$

$$= -5\cos(\ln x) + \sin(\ln x) + 2\cos(\ln x) - 3\sin(\ln x)$$

$$= -3\cos(\ln x) - 2\sin(\ln x)$$

$$= -\left[3\cos(\ln x) + 2\sin(\ln x)\right]$$

$$= -y \qquad \therefore \quad A = -1$$

IYGB - SYNOPTIC PAPER I - QUESTION 12

ALTERNATIVE APPROACH

$$y = 3\cos(\ln x) + 2\sin(\ln x)$$

$$\frac{dy}{dx} = -3\sin(\ln x) \times \frac{1}{x} + 2\cos(\ln x) \times \frac{1}{x}$$

MULTIPLY ACROSS & DIFFERENTIATE L.H.S WITH RESPECT TO x

$$x\frac{dy}{dx} = -3\sin(\ln x) + 2\cos(\ln x)$$

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] = \frac{d}{dx}\left[-3\sin(\ln x) + 2\cos(\ln x)\right]$$

$$1 \times \frac{dy}{dx} + x\frac{d^2y}{dx^2} = -3\cos(\ln x) \times \frac{1}{x} - 2\sin(\ln x) \times \frac{1}{x}$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}\left[3\cos(\ln x) + 2\sin(\ln x)\right]$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\left[3\cos(\ln x) + 2\sin(\ln x)\right]$$

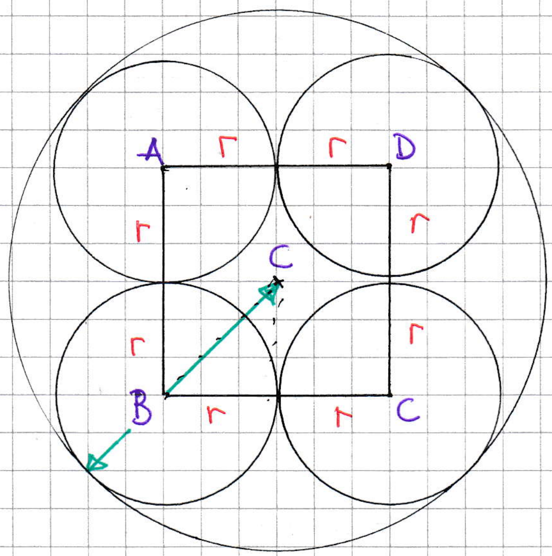
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$\therefore A = -1$

AS BEFORE

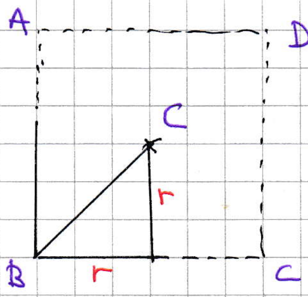
IYGB - SYNOPTIC PAPER I - QUESTION 13

STARTING WITH A DIAGRAM



- LET THE SIDE OF THE SQUARE BE $2r$, WHERE r IS THE RADIUS OF THE SMALLER CIRCLES
- TOTAL AREA OF THE 4 SMALL CIRCLES IS $4\pi r^2$
- LET C BE THE CENTRE OF THE LARGER CIRCLE, WHICH IS ALSO THE CENTRE OF ABCD

BY PYTHAGORAS



$$|BC|^2 = r^2 + r^2$$
$$|BC|^2 = 2r^2$$
$$|BC| = \sqrt{2}\,r$$

RADIUS OF THE LARGER CIRCLE IS $r + \sqrt{2}r$ (MARKED IN GREEN)

AREA OF BIG CIRCLE IS
$$\pi(r + \sqrt{2}r)^2$$
$$\pi(1+\sqrt{2})^2 r^2$$
$$\pi(1 + 2\sqrt{2} + 2)r^2$$
$$\pi(3 + 2\sqrt{2})r^2$$

IYGB - SYNOPTIC PAPER I - QUESTION 13

THE REQUIRED RATIO IS

$$\frac{\text{AREA OF 4 SMALL CIRCLES}}{\text{AREA OF BIG CIRCLE}} = \frac{4\cancel{\pi}\cancel{r^2}}{\cancel{\pi}(3+2\sqrt{2})\cancel{r^2}}$$

$$= \frac{4}{3+2\sqrt{2}}$$

$$= \frac{4(3-2\sqrt{2})}{(3+2\sqrt{2})(3-2\sqrt{2})}$$

$$= \frac{12-8\sqrt{2}}{9-6\sqrt{2}+6\sqrt{2}-8}$$

$$= \frac{12-8\sqrt{2}}{1}$$

IE RATIO OF

$12-8\sqrt{2} : 1$

AS REQUIRED

IYGB - SYNOPTIC PAPER I - QUESTION 14

TIDY THE EXPRESSION INTO INDICIAL FORM

$$\Rightarrow y = \frac{1}{3\sqrt{x}}\left[\frac{2}{x} - 3\right]$$

$$\Rightarrow y = \frac{1}{3}x^{-\frac{1}{2}}\left[2x^{-1} - 3\right]$$

$$\Rightarrow y = \frac{2}{3}x^{-\frac{3}{2}} - x^{-\frac{1}{2}}$$

FIND THE GRADIENT FUNCTION

$$\Rightarrow \frac{dy}{dx} = -x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$$

DECREASING $\Rightarrow \frac{dy}{dx} < 0$

$$\Rightarrow -x^{-\frac{5}{2}} + \frac{1}{2}x^{-\frac{3}{2}} < 0$$

$$\Rightarrow \frac{1}{2}x^{-\frac{3}{2}} < x^{-\frac{5}{2}}$$

$$\Rightarrow x^{-\frac{3}{2}} < 2x^{-\frac{5}{2}}$$

$$\Rightarrow \frac{1}{x^{3/2}} < \frac{2}{x^{5/2}}$$

AS $x > 0$ WE MAY MULTIPLY THE INEQUALITY THROUGH WITHOUT REVERSING DIRECTION

$$\Rightarrow \frac{x^{5/2}}{x^{3/2}} < 2$$

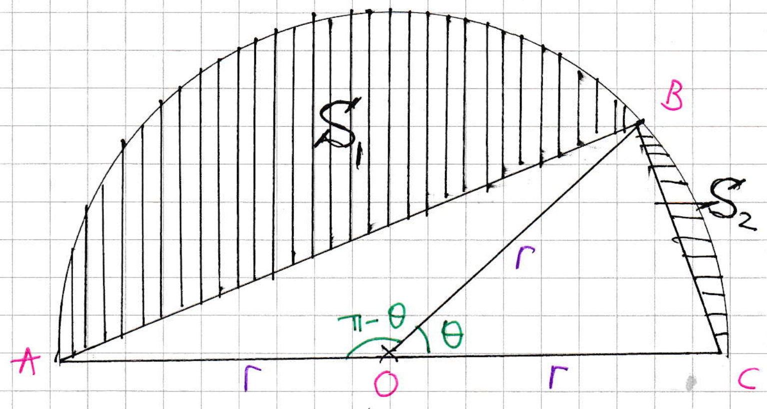
$$\Rightarrow x < 2$$

BUT SINCE $x > 0$

$$0 < x < 2$$

IYGB - SYNOPTIC PAPER I - QUESTION 15

PROCEED AS FOLLOWS



AREA OF SECTOR $\overset{\triangleright}{BOC}$

$= \frac{1}{2}r^2\theta$

AREA OF TRIANGLE $\overset{\triangle}{BOC}$

$= \frac{1}{2}r^2\sin\theta$

AREA OF S_2 $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

AREA OF SECTOR $\overset{\triangleright}{AOB}$

$= \frac{1}{2}r^2(\pi-\theta)$

AREA OF TRIANGLE $\overset{\triangle}{AOB}$

$= \frac{1}{2}r^2\sin(\pi-\theta) = \frac{1}{2}r^2\sin\theta$

SINCE $\sin\theta \equiv \sin(\pi-\theta)$

AREA OF S_1 $= \frac{1}{2}r^2(\pi-\theta) - \frac{1}{2}r^2\sin\theta$

FINALLY WE ARE GIVEN THAT

"S_1 AREA" = 4 "S_2 AREA"

$\frac{1}{2}r^2(\pi-\theta) - \frac{1}{2}r^2\sin\theta = 4\left[\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta\right]$

DIVIDE BY $\frac{1}{2}r^2$

$\pi-\theta - \sin\theta = 4\left[\theta - \sin\theta\right]$

$\pi-\theta - \sin\theta = 4\theta - 4\sin\theta$

$\pi + 3\sin\theta = 5\theta$

AS REQUIRED

IYGB - SYNOPTIC PAPER I - QUESTION 16

$$\boxed{\text{IF } f(x) = \frac{x-2}{x+2} \quad \text{THEN} \quad f(x+h) = \frac{(x+h)-2}{(x+h)+2}}$$

$$f(x+h) - f(x) = \frac{x+h-2}{x+h+2} - \frac{x-2}{x+2} = \frac{(x+2)(x-2+h) - (x-2)(x+2+h)}{(x+2)(x+h+2)}$$

$$= \frac{\cancel{x^2-4} + h(x+2) - [\cancel{x^2-4} + h(x-2)]}{(x+2)(x+h+2)}$$

$$= \frac{hx + 2h - hx + 2h}{(x+2)(x+h+2)} = \frac{4h}{(x+2)(x+h+2)}$$

FROM THE DEFINITION OF THE DERIVATIVE AS A LIMIT

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right] = \lim_{h \to 0}\left[[f(x+h) - f(x)] \div h\right]$$

$$= \lim_{h \to 0}\left[\frac{4\cancel{h}}{(x+2)(x+h+2)} \times \frac{1}{\cancel{h}}\right]$$

$$= \lim_{h \to 0}\left[\frac{4}{(x+2)(x+h+2)}\right]$$

$$= \frac{4}{(x+2)^2}$$

IYGB - SYNOPTIC PAPER I - QUESTION 17

LET THE FOUR CONSECUTIVE POSITIVE INTEGERS ARE $n, n+1, n+2, n+3$

THEN WE HAVE

$$\sqrt{n(n+1)(n+2)(n+3)+1} = \sqrt{(n^2+n)(n^2+5n+6)+1}$$

$$= \sqrt{\begin{matrix} n^4+5n^3+6n^2 \\ \quad n^3+5n^2+6n+1 \end{matrix}}$$

$$= \sqrt{n^4+6n^3+11n^2+6n+1}$$

NOW THIS MUST BE A PERFECT SQUARE

$$n^4+6n^3+11n^2+6n+1 \equiv (n^2+An+1)^2$$

$$\equiv n^4+A^2n^2+1+2n^2+2An^3+2An$$

$$\equiv n^4+2An^3+(A^2+2)n^2+2An+1$$

$$\therefore A=3$$

$$\therefore \sqrt{n(n+1)(n+2)(n+3)+1} = \sqrt{(n^2+3n+1)^2} = n^2+3n+1$$

AS REQUIRED

NOTE MODS ARE NOT REALLY NEEDED HERE

IYGB - SYNOPTIC PAPER I - QUESTION 18

WORK WITH 3 DERIVATIVES AS FOLLOWS

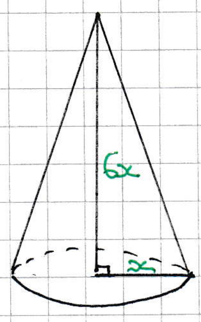
$$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dA} \times \frac{dA}{dt}$$

$$\frac{dV}{dt} = \cancel{6\pi}x^2 \times \frac{1}{\cancel{2\pi}x} \times 0.25$$

$$\frac{dV}{dt} = \frac{3}{4}x$$

$$\left.\frac{dV}{dt}\right|_{x=2.5} = \frac{3}{4} \times \frac{5}{2} = \frac{15}{8} = 1.875 \text{ cm}^3\text{s}^{-1}$$



BASE AREA

$A = \pi x^2$

$\frac{dA}{dx} = 2\pi x$

TOTAL VOLUME

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi x^2(6x)$

$V = 2\pi x^3$

$\frac{dV}{dx} = 6\pi x^2$

ALTERNATIVE APPROACH

$$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

$$\frac{dV}{dt} = \frac{6A^2}{\pi V} \times 0.25$$

$$\frac{dV}{dt} = \frac{6(\pi x^2)^2}{\pi(2\pi x^3)} \times \frac{1}{4}$$

$$\frac{dV}{dt} = \frac{6\pi^2 x^4}{8\pi^2 x^3} = \frac{3}{4}x$$

$$\left.\frac{dV}{dt}\right|_{x=2.5} = \frac{3}{4} \times 2.5 = 1.875 \text{ cm}^3\text{s}^{-1}$$

$V^2 = \frac{4}{\pi}A^3$

$2V\frac{dV}{dA} = \frac{12}{\pi}A^2$

$\frac{dV}{dA} = \frac{6A^2}{\pi V}$

- $A = \pi x^2$
- $V = 2\pi x^3$ (AS ABOVE)
- $A^3 = \pi^3 x^6$
- $V^2 = 4\pi^2 x^6$

DIVIDING

$\frac{V^2}{A^3} = \frac{4\pi^2 x^6}{\pi^3 x^6}$

$\frac{V^2}{A^3} = \frac{4}{\pi}$

$V^2 = \frac{4}{\pi}A^3$

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REWRITE THE EQUATION AS A QUADRATIC IN 3^t

$$\Rightarrow 3^{t+1} = 6 + 3^{2t-1}$$
$$\Rightarrow 3^t \times 3^1 = 6 + 3^{2t} \times 3^{-1}$$
$$\Rightarrow 3(3^t) = 6 + (3^t)^2 \times \frac{1}{3}$$
$$\Rightarrow 3a = 6 + \frac{1}{3}a^2 \qquad [\text{WHERE } a = 3^t]$$

SOLVE THE QUADRATIC IN a

$$\Rightarrow 9a = 18 + a^2$$
$$\Rightarrow 0 = a^2 - 9a + 18$$
$$\Rightarrow (a-3)(a-6) = 0$$
$$\Rightarrow a = \begin{cases} 3 \\ 6 \end{cases}$$
$$\Rightarrow 3^t = \begin{cases} 3 \\ 6 \end{cases}$$

BY INSPECTION FOR $3^t = 3$ & USING LOGS FOR $3^t = 6$

EITHER $t = 1$ $\qquad$ OR

$$3^t = 6$$
$$\log 3^t = \log 6$$
$$t \log 3 = \log 6$$
$$t = \frac{\log 6}{\log 3}$$
$$t \approx 1.63 \quad \text{3 s.f.}$$

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a) USING THE STANDARD METHOD

$$f(-x) = \ln\left[\frac{1+(-x)}{1-(-x)}\right] = \ln\left(\frac{1-x}{1+x}\right) = \ln\left(\frac{1+x}{1-x}\right)^{-1}$$

$$= -\ln\left(\frac{1+x}{1-x}\right) = -f(x)$$

AS $f(-x) = -f(x)$ THE FUNCTION IS ODD

b) DIFFERENTIATING AFTER MANIPULATING

$$f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$f'(x) = \frac{1}{1+x} + \frac{1}{1-x}$$

$$f'(x) = \frac{(1\cancel{-x}) + (1\cancel{+x})}{(1+x)(1-x)}$$

$$f'(x) = \frac{2}{1-x^2}$$

CHECKING $f'(-x)$

$$f'(-x) = \frac{2}{1-(-x)^2} = \frac{2}{1-x^2} = f'(x)$$

AS $f'(-x) = f'(x)$, $f'(x)$ IS EVEN

c) WRITE $f(x)$ AS y & REARRANGE

$$y = \ln\left(\frac{1+x}{1-x}\right) \implies e^y = \frac{1+x}{1-x}$$

$$\implies e^y(1-x) = 1+x$$

$$\implies e^y - xe^y = 1+x$$

$$\implies e^y - 1 = x + xe^y$$

$$\implies x(1+e^y) = e^y - 1$$

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$$\Rightarrow x = \frac{e^y - 1}{e^y + 1}$$

$$\therefore f^{-1}(x) = \frac{e^x - 1}{e^x + 1}$$

d) $$\int_0^{\ln 3} f^{-1}(x)\,dx = \int_0^{\ln 3} \frac{e^x - 1}{e^x + 1}\,dx$$

$u = e^x + 1$

$\frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$e^x = u - 1$

$x = 0,\ u = 2$

$x = \ln 3,\ u = 4$

$$= \int_2^4 \frac{e^x - 1}{u}\left(\frac{du}{e^x}\right) = \int_2^4 \frac{u-2}{u(u-1)}\,du$$

... PARTIAL FRACTIONS ...

$$\frac{u-2}{u(u-1)} \equiv \frac{A}{u} + \frac{B}{u-1}$$

$$\boxed{u - 2 \equiv A(u-1) + Bu}$$

- IF $u = 0$: $-2 = -A$, $A = 2$
- IF $u = 1$: $-1 = B$, $B = -1$

$$= \int_2^4 \frac{2}{u} - \frac{1}{u-1}\,du = \Big[2\ln|u| - \ln|u-1|\Big]_2^4$$

$$= (2\ln 4 - \ln 3) - (2\ln 2 - \cancel{\ln 1}) = 2\ln 4 - \ln 3 - 2\ln 2$$

$$= \ln 16 - \ln 3 - \ln 4 = \ln\left(\frac{16}{3 \times 4}\right) = \ln\left(\frac{16}{12}\right)$$

$$= \ln\left(\frac{4}{3}\right)$$

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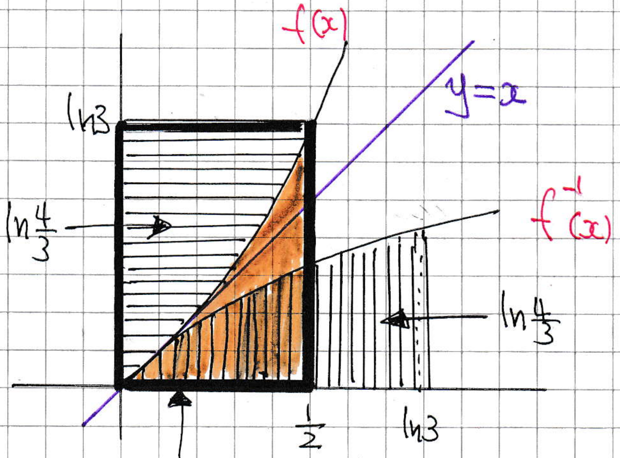
e) LOOKING AT THE DIAGRAM BELOW



RECTANGLE $= \frac{1}{2}\ln 3$

$\therefore$ REQUIRED AREA $= \frac{1}{2}\ln 3 - \ln\frac{4}{3}$

$= \frac{1}{2}\ln 3 - (\ln 4 - \ln 3)$

$= \frac{1}{2}\ln 3 - 2\ln 2 + \ln 3$

$= \frac{3}{2}\ln 3 - \ln 4$

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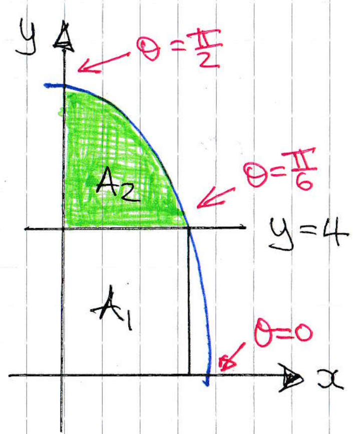
METHOD 1 – PARAMETRIC INTEGRATION IN x



- $x = \cos\theta$ • $y = 8\sin\theta$

$y = 4$
$4 = 8\sin\theta$
$\sin\theta = \frac{1}{2}$
$\theta = \frac{\pi}{6}$

• AREA $(A_1 + A_2)$ $= \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$

$= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} (8\sin\theta)(-\sin\theta)\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{6}} + 8\sin^2\theta\,d\theta$

$= \int_{\frac{\pi}{6}}^{\pi/2} 8\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 4\cos 2\theta\,d\theta$

$= \left[4\theta - 2\sin 2\theta\right]_{\pi/6}^{\pi/2} = (2\pi - 0) - \left(\frac{2\pi}{3} - \sqrt{3}\right)$

$= \frac{4}{3}\pi + \sqrt{3}$

• SUBTRACTING THE RECTANGLE A_1

WHEN $\theta = \frac{\pi}{6}$ $x = \frac{\sqrt{3}}{2}$

$A_1 = \frac{\sqrt{3}}{2} \times 4 = 2\sqrt{3}$

• REQUIRED AREA $= \frac{4}{3}\pi + \sqrt{3} - 2\sqrt{3}$

$= \frac{4}{3}\pi - \sqrt{3}$

METHOD 2 – PARAMETRIC INTEGRATION IN y

• AREA $A_2 = \int_{y_1}^{y_2} x(y)\,dy = \int_{\theta_1}^{\theta_2} x(\theta)\frac{dy}{d\theta}\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos\theta(8\cos\theta)\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\cos^2\theta\,d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 + 4\cos 2\theta\,d\theta$

$= \left[4\theta + 2\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = (2\pi + 0) - \left(\frac{2\pi}{3} + \sqrt{3}\right)$

$= \frac{4\pi}{3} - \sqrt{3}$

AS ABOVE

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THE FIRST SUMMATION (INNER NEST) IS GEOMETRIC

E.g $\sum_{r=1}^{k} 2^r = 2^1 + 2^2 + 2^3 + \dots + 2^k$

SUM OF THE FIRST k TERMS

HENCE WE HAVE

$$\sum_{k=1}^{n}\left[\sum_{r=1}^{k} 2^r\right] = \sum_{k=1}^{n}\left[\frac{2(2^k-1)}{2-1}\right]$$

$$= \sum_{k=1}^{n}\left[2(2^k-1)\right]$$

$$= 2\sum_{k=1}^{n}\left[2^k - 1\right]$$

$$= 2\sum_{k=1}^{n}\left[2^k\right] - 2\sum_{k=1}^{n} 1$$

REPEATING AS THE FIRST IS THE VERY SAME G.P AS ABOVE

$$= 2\left[\frac{2(2^n-1)}{2-1}\right] - 2 \times n$$

$$= 2\left[2(2^n-1)\right] - 2n$$

$$= 4(2^n-1) - 2n$$

$$= 4 \times 2^n - 4 - 2n$$

$$= 2^{n+2} - 2n - 4$$

AS REQUIRED

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CHANGE INTO BASE e

$$\int_1^e \log_{10} x \, dx = \int_1^e \frac{\log_e x}{\log_e 10} \, dx$$

$$\boxed{\log_a b \equiv \frac{\log_c b}{\log_c a}}$$

$$= \int_1^e \frac{\ln x}{\ln 10} \, dx$$

$$= \int_1^e \frac{1}{\ln 10} (\ln x) \, dx$$

INTEGRATION BY PARTS FOLLOWS NOTING THAT $\frac{1}{\ln 10}$ IS A CONSTANT

$\ln x$	$\frac{1}{x}$
$\frac{1}{\ln 10} x$	$\frac{1}{\ln 10}$

$$= \left[\frac{\ln x}{\ln 10} x \right]_1^e - \int_1^e \frac{1}{\ln 10} \cancel{x} \times \frac{1}{\cancel{x}} \, dx$$

$$= \left[\frac{x \ln x}{\ln 10} \right]_1^e - \int_1^e \frac{1}{\ln 10} \, dx$$

$$= \left[\frac{x \ln x}{\ln 10} - \frac{1}{\ln 10} x \right]_1^e$$

$$= \left(\frac{e \ln e}{\ln 10} - \frac{e}{\ln 10} \right) - \left(\cancel{\frac{\ln 1}{\ln 10}} - \frac{1}{\ln 10} \right)$$

$$= \left(\frac{e}{\ln 10} - \frac{e}{\ln 10} \right) + \frac{1}{\ln 10}$$

$$= \frac{1}{\ln 10}$$

AS REQUIRED