IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper J

Difficulty Rating: 4.1125/7417

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. (**2**).
There are 21 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

The equation of a curve is given implicitly by

$$2\ln y = x\ln x, \quad x, y \in \mathbb{R} \quad x, y > 0.$$

Find the exact value of the gradient at the point on the curve where $x = 4$. **(6)**

Question 2

$$y = \frac{7x+2}{(x-2)(x+2)(2x+1)}, \ x \neq \pm 2, \ x \neq -\frac{1}{2}.$$

Find the exact value of $\frac{dy}{dx}$ at $x = -1$. **(9)**

Question 3

The sum, S_n, of the first n terms of an arithmetic series is given by

$$S_n = 2n(4n-7).$$

Find the fifth term of the series. **(4)**

Question 4

$$f(x) = \frac{(1-x)^2}{\sqrt{1+2x}}, \quad |x| < \frac{1}{2}$$

Show that if x is small, then

$$f(x) \approx 1 - 3x + \frac{9}{2}x^2 - \frac{13}{2}x^3.$$ **(8)**

Question 5

Linda is walking on a long straight horizontal road in a Northern direction.

When Linda reaches a point A on this road, a tree T is observed on a bearing of $30°$.

When Linda walks a further distance of 200 m from the point A to the point B on this road, T is now observed on a bearing of $60°$.

a) Determine the shortest distance of T from the road. **(3)**

Linda walks further North to some point D, so that the distance DT is 180 m.

b) Calculate the two possible values for the distance AD. **(8)**

Question 6

The function f satisfies

$$f : x \mapsto \frac{3x+1}{x+4}, \ x \in \mathbb{R}, \ x > -4.$$

a) Find an expression for $f^{-1}(x)$ in its simplest form. **(3)**

b) Determine the domain and the range of $f^{-1}(x)$. **(3)**

The function g is given by

$$g : x \mapsto e^{x} - 3, \ x \in \mathbb{R}.$$

c) Solve the equation

$$fg(x) = \frac{4}{5},$$

giving exact answers in terms of $\ln 2$. **(5)**

Question 7

The number of bacteria N present in a culture at time t hours, is modelled by the equation

$$N = Ae^{kt}, \ t \geq 0.$$

At the instant when $t = \ln 64$, there are 1200 bacteria present in the culture and bacteria are increasing at the rate of 200 bacteria per hour.

a) Find the value of A and the value of k. **(6)**

The time T it takes for the bacteria to triple in number is constant.

b) Find the exact value of T. **(4)**

Question 8

The points A, B and C have coordinates $(-3,0)$, $(-1,6)$ and $(11,2)$, respectively.

a) Show clearly that

$$\measuredangle ABC = 90°.$$ **(3)**

The points A, B and C lie on the circumference of a circle centred at the point D.

b) Find an equation for this circle in the form

$$x^2 + y^2 + ax + by + c = 0,$$

where a, b and c, are constants to be found. **(7)**

Question 9

By using the substitution $x = -\frac{1}{2} + \frac{1}{2}\sin\theta$, or otherwise, find the exact value of

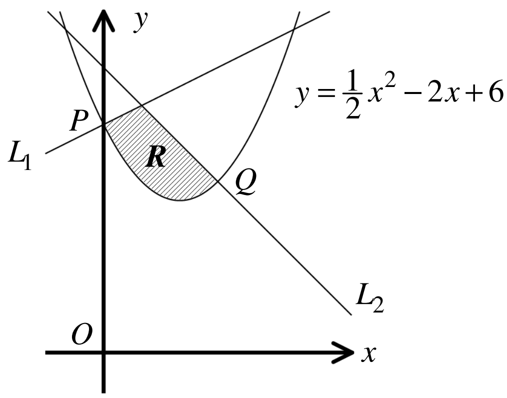
$$\int_{-\frac{1}{4}}^{0} \frac{3}{\sqrt{-x(x+1)}} \, dx.$$ **(11)**

Question 10

Solve in **degrees** the trigonometric equation

$$4\tan(\theta+60)\tan(\theta-60)=\sec^2\theta-16,\ 0°\le\theta<180°.$$ **(12)**

Question 11



The figure above shows the graph of the curve C with equation

$$y=\frac{1}{2}x^2-2x+6.$$

The point P is the point where C meets the y axis so that the straight line L_1 is the normal to C at P.

a) Find an equation for L_1. **(4)**

The point $Q\left(3,\frac{9}{2}\right)$ lies on C and the straight line L_2 is the normal to C at Q.

The finite region R, shown shaded in the figure above, is bounded by L_1, L_2 and C.

b) Find the area of R. **(10)**

Question 12

It is given that

$$\overrightarrow{AP} + 4\overrightarrow{BP} + 3\overrightarrow{PC} = \vec{0}.$$

Show that

$$\overrightarrow{AP} = \frac{1}{2}\left[\overrightarrow{AB} - 3\overrightarrow{BC}\right].$$ **(6)**

Question 13

A curve has equation $y = f(x)$ given by

$$f(x) = 2 + \frac{1}{2x-1}, \quad x \neq \frac{1}{2}.$$

a) Express $f(x)$ as a single simplified fraction. **(2)**

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\frac{1}{x} \xrightarrow{T_1} \frac{1}{x-1} \xrightarrow{T_2} \frac{1}{2x-1} \xrightarrow{T_3} 2 + \frac{1}{2x-1}.$$

b) Describe geometrically the transformations T_1, T_2 and T_3. **(3)**

c) Hence sketch the graph of $f(x)$.

Indicate clearly any asymptotes and the coordinates of any intersections with the coordinate axes. **(5)**

d) Find the coordinates of the point of intersection of $f(x)$ and the line $y = 3$. **(2)**

Question 14

The points $A(0,3k)$ and $B(0,k)$ are given, where k is a non zero constant.

The point P lies on the straight line with equation $y = x$, so that both straight lines, AP and BP, have negative gradient.

The straight line through A and P meets the x axis at x_1 and the straight line through B and P meets the x axis at x_2.

Show that $\dfrac{1}{x_1} - \dfrac{1}{x_2} = \dfrac{2}{3k}$. **(10)**

Question 15

Differentiate $\dfrac{1}{2-x}$ from first principles. **(6)**

Question 16

In a laboratory a dangerous chemical is stored in a cylindrical drum of height 160 cm which is initially full.

One day the drum was found leaking and when this was first discovered, the level of the chemical had dropped to 100 cm, and at that instant the level of the chemical was dropping at the rate of 0.25 cm per minute.

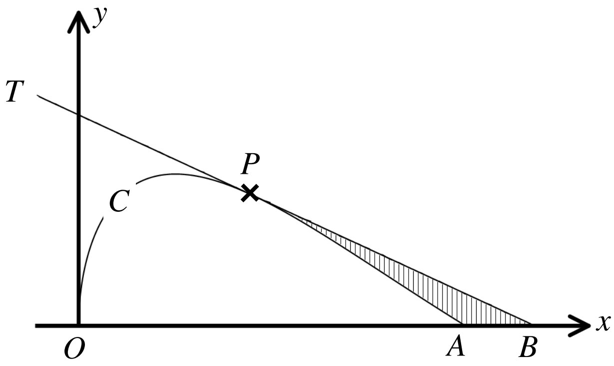
In order to assess the contamination level in the laboratory, it is required to find the length of time that the leaking has been taking place.

It is assumed that the rate at which the height of the chemical is dropping is proportional to the square root of its height.

a) Form a suitable differential equation to model the above problem, where the time, in minutes, is measured from the instant that the leaking was discovered. **(2)**

b) Find a solution of the differential equation and use it to calculate, in hours and minutes, for how long the leaking has been taking place. **(10)**

Question 17



The figure above shows the curve with parametric equations

$$x = t^2, \ y = \sin t, \ 0 \le t \le \pi .$$

The curve crosses the x axis at the origin O and at the point A.

The point P lies on the curve where $t = \frac{2}{3}\pi$.

The straight line T is a tangent to the curve at P.

Show that the area of the finite region bounded by the curve, the tangent T and the x axis, shown shaded in the figure above, is

$$\frac{1}{3}\left(3\sqrt{3} - \pi\right). \qquad \textbf{(15)}$$

Question 18

$$f(k) = k^3 + 2k, \quad k \in \mathbb{N}.$$

Without using proof by induction, show that $f(k)$ is always a multiple of 3. **(6)**

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Question 19

Evaluate showing clearly your method

$$\sum_{n=1}^{\infty} \frac{3^n - 2}{4^{n+1}}. \qquad (8)$$

Question 20

Show clearly, without approximating and without using any calculating aid, that

a) $\sqrt{6+2\sqrt{6}} > \sqrt{3}+\sqrt{2}$. **(3)**

b) $\sqrt[3]{3} > \sqrt{2}$. **(3)**

c) $\sqrt{2}-1 > \sqrt{3}-\sqrt{2}$. **(3)**

Question 21

Solve the following trigonometric equation for $0 \le \theta < 360°$

$$2 + 4\cos^2\theta = 7\cos\theta\sin\theta. \qquad (10)$$
