OCR

Oxford Cambridge and RSA

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Insert

Practice Paper – Set 2

Time allowed: 2 hours

INFORMATION FOR CANDIDATES

- This insert contains the article for Section B.
- This document consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER/INVIGILATOR

- Do not send this Insert for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

Solving cubic equations

Modern scientific calculators will solve cubic equations, giving the real roots in decimal form. For example, a calculator gives the root of $x^3+6x^2+12x+13=0$ as -3.71 correct to 2 decimal places. The root is the x-coordinate of the point where the curve $y=x^3+6x^2+12x+13$ crosses the x-axis, as shown in Fig. C1. Drawing the curve indicates that the equation has only one real root.

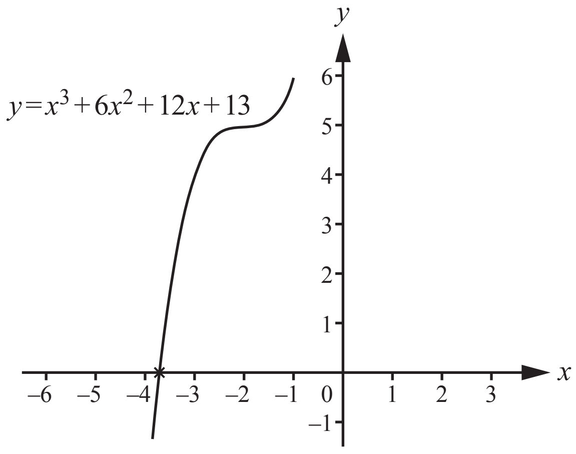

Fig. C1

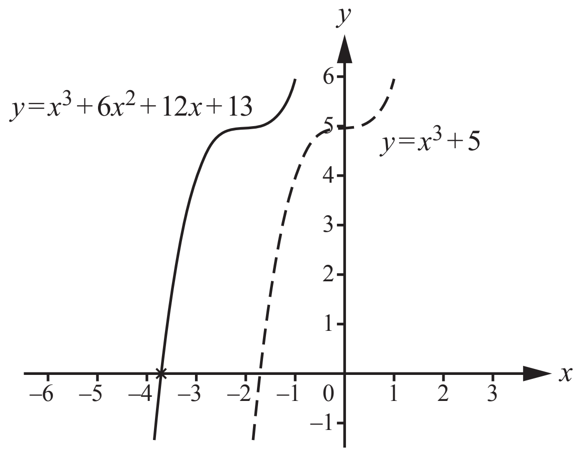

Fig. C2

A general method for solving cubic equations

The general formula for solving the quadratic equation $ax^2+bx+c=0$ is well known; this is based on methods known to the Babylonians 2400 years ago. A method for solving cubic equations came much later.

Scipione dal Ferro first solved cubic equations of the form $x^3+mx=n$ around 1515 in Italy but the method did not become widely known until 30 years later. Any cubic equation can be transformed into the form $x^3+mx=n$; this process is sometimes known as 'depressing' the cubic equation.

Depressing a cubic equation

In the equation $x^3+6x^2+12x+13=0$, replace x by $(x-2)$ to give

$$(x-2)^3+6(x-2)^2+12(x-2)+13=0.$$

This simplifies to $x^3+5=0$. The curve $y=x^3+5$ is a translation of the curve $y=x^3+6x^2+12x+13$, as shown in Fig. C2.

The depressed equation, $x^3+5=0$, has one real root, $x=-\sqrt[3]{5}$. Hence the original equation has one real root, $x=-\sqrt[3]{5}-2$.

In general, for the cubic equation $ax^3+bx^2+cx+d=0$, replacing x by $\left(x-\dfrac{b}{3a}\right)$ will give a cubic equation with no x^2 term.

Vieta's substitution

In the example above, the depressed equation was easy to solve but this is not always the case. There are several ways of solving a cubic equation of the form $x^3 + mx = n$. One of them is by using Vieta's substitution, $x = y - \frac{m}{3y}$; this is not the method used by dal Ferro, but the working is simpler. Vieta's substitution is illustrated below for the equation $x^3 + 3x = 5$.

The required substitution in this case is $x = y - \frac{1}{y}$. This gives x^3 in terms of y as follows.

$$\begin{aligned} x^3 &= \left(y - \frac{1}{y}\right)^3 \\ &= y^3 - 3y^2\left(\frac{1}{y}\right) + 3y\left(\frac{1}{y}\right)^2 - \left(\frac{1}{y}\right)^3 \\ &= y^3 - 3y + \frac{3}{y} - \frac{1}{y^3} \end{aligned}$$

Substituting for both x^3 and x into the equation $x^3 + 3x = 5$ gives

$$y^3 - 3y + \frac{3}{y} - \frac{1}{y^3} + 3\left(y - \frac{1}{y}\right) = 5,$$

which simplifies to

$$y^3 - \frac{1}{y^3} = 5.$$

Multiplying through by y^3 gives the following quadratic equation in y^3.

$$(y^3)^2 - 5y^3 - 1 = 0$$

Solving this quadratic equation gives $y^3 = \frac{5 \pm \sqrt{29}}{2}$.

Solving the original cubic equation

There are two possible values of y but substituting back into $x = y - \frac{1}{y}$ results in the same value of x for each of them. This value is $x = 1.15$, correct to 2 decimal places, and this is the root of the depressed equation. The root of the original equation can now be found.

Going beyond the real numbers

If the original cubic equation has three real roots, the quadratic equation in y^3 obtained by using Vieta's substitution does not have real roots. However, by using techniques beyond A Level Mathematics, it is still possible to find the roots of the original cubic equation.